

# Mathematics

## Calculus 3208

*Curriculum Guide 2018*



Education and Early Childhood Development



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# INTRODUCTION

## Background

*The curriculum guide communicates high expectations for students.*

The Mathematics curriculum guides for Newfoundland and Labrador have been derived from *The Common Curriculum Framework for 10-12 Mathematics: Western and Northern Canadian Protocol*, January 2008. The curriculum guide for Calculus 3208 incorporates this conceptual framework. It includes the general outcomes, specific outcomes and achievement indicators. It also includes suggestions for teaching and learning, suggested assessment strategies, and an identification of the associated resource match between the curriculum and authorized, as well as recommended, resource materials.

Calculus 3208 was first implemented in 2013.

## Beliefs About Students and Mathematics

*Mathematical understanding is fostered when students build on their own experiences and prior knowledge.*

Students are curious, active learners with individual interests, abilities and needs. They come to classrooms with varying knowledge, life experiences and backgrounds. A key component in developing mathematical literacy is making connections to these backgrounds and experiences.

Students learn by attaching meaning to what they do, and they need to construct their own meaning of mathematics. This meaning is best developed when learners encounter mathematical experiences that proceed from the simple to the complex and from the concrete to the abstract. Through the use of manipulatives and a variety of pedagogical approaches, teachers can address the diverse learning styles, cultural backgrounds and developmental stages of students, and enhance within them the formation of sound, transferable mathematical understandings. Students at all levels benefit from working with a variety of materials, tools and contexts when constructing meaning about new mathematical ideas. Meaningful student discussions provide essential links among concrete, pictorial and symbolic representations of mathematical concepts.

The learning environment should value and respect the diversity of students' experiences and ways of thinking, so that students feel comfortable taking intellectual risks, asking questions and posing conjectures. Students need to explore problem-solving situations in order to develop personal strategies and become mathematically literate. They must come to understand that it is acceptable to solve problems in a variety of ways and that a variety of solutions may be acceptable.

# Program Design and Components

## Affective Domain

*To experience success, students must learn to set achievable goals and assess themselves as they work toward these goals.*

A positive attitude is an important aspect of the affective domain and has a profound impact on learning. Environments that create a sense of belonging, encourage risk taking and provide opportunities for success help develop and maintain positive attitudes and self-confidence within students. Students with positive attitudes toward learning mathematics are likely to be motivated and prepared to learn, participate willingly in classroom activities, persist in challenging situations and engage in reflective practices.

Teachers, students and parents need to recognize the relationship between the affective and cognitive domains, and attempt to nurture those aspects of the affective domain that contribute to positive attitudes. To experience success, students must learn to set achievable goals and assess themselves as they work toward these goals.

Striving toward success and becoming autonomous and responsible learners are ongoing, reflective processes that involve revisiting, assessing and revising personal goals.

## Goals For Students

*Mathematics education must prepare students to use mathematics confidently to solve problems.*

The main goals of mathematics education are to prepare students to:

- use mathematics confidently to solve problems
- communicate and reason mathematically
- appreciate and value mathematics
- make connections between mathematics and its applications
- commit themselves to lifelong learning
- become mathematically literate adults, using mathematics to contribute to society.

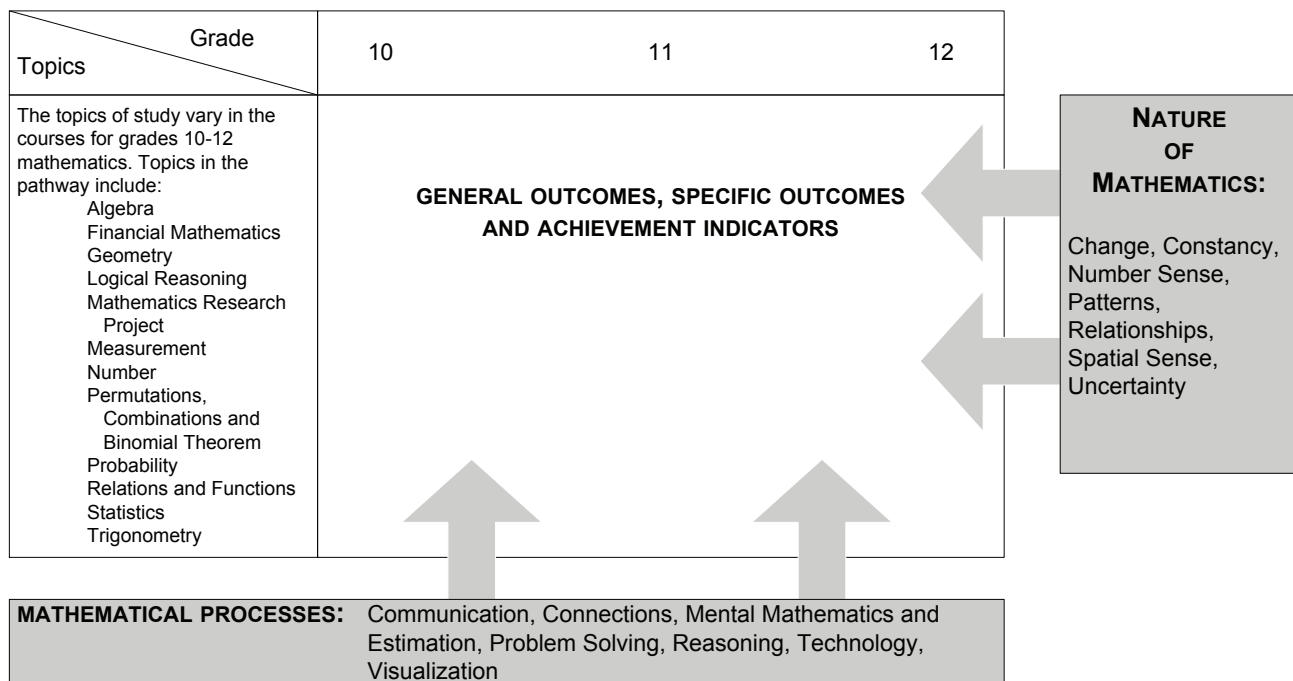
Students who have met these goals will:

- gain understanding and appreciation of the contributions of mathematics as a science, philosophy and art
- exhibit a positive attitude toward mathematics
- engage and persevere in mathematical tasks and projects
- contribute to mathematical discussions
- take risks in performing mathematical tasks
- exhibit curiosity.



# CONCEPTUAL FRAMEWORK FOR 10-12 MATHEMATICS

The chart below provides an overview of how mathematical processes and the nature of mathematics influence learning outcomes.



## Mathematical Processes

- *Communication [C]*
- *Connections [CN]*
- *Mental Mathematics and Estimation [ME]*
- *Problem Solving [PS]*
- *Reasoning [R]*
- *Technology [T]*
- *Visualization [V]*

There are critical components that students must encounter in a mathematics program in order to achieve the goals of mathematics education and embrace lifelong learning in mathematics.

Students are expected to:

- communicate in order to learn and express their understanding
- connect mathematical ideas to other concepts in mathematics, to everyday experiences and to other disciplines
- demonstrate fluency with mental mathematics and estimation
- develop and apply new mathematical knowledge through problem solving
- develop mathematical reasoning
- select and use technologies as tools for learning and for solving problems
- develop visualization skills to assist in processing information, making connections and solving problems.

This curriculum guide incorporates these seven interrelated mathematical processes that are intended to permeate teaching and learning.

**Communication [C]**

*Students must be able to communicate mathematical ideas in a variety of ways and contexts.*

Students need opportunities to read about, represent, view, write about, listen to and discuss mathematical ideas. These opportunities allow students to create links between their own language and ideas, and the formal language and symbols of mathematics.

Communication is important in clarifying, reinforcing and modifying ideas, attitudes and beliefs about mathematics. Students should be encouraged to use a variety of forms of communication while learning mathematics. Students also need to communicate their learning using mathematical terminology.

Communication helps students make connections among concrete, pictorial, symbolic, oral, written and mental representations of mathematical ideas.

**Connections [CN]**

*Through connections, students begin to view mathematics as useful and relevant.*

Contextualization and making connections to the experiences of learners are powerful processes in developing mathematical understanding. When mathematical ideas are connected to each other or to real-world phenomena, students begin to view mathematics as useful, relevant and integrated.

Learning mathematics within contexts and making connections relevant to learners can validate past experiences and increase student willingness to participate and be actively engaged.

The brain is constantly looking for and making connections. “Because the learner is constantly searching for connections on many levels, educators need to *orchestrate the experiences* from which learners extract understanding ... Brain research establishes and confirms that multiple complex and concrete experiences are essential for meaningful learning and teaching” (Caine and Caine, 1991, p.5).

## Mental Mathematics and Estimation [ME]

*Mental mathematics and estimation are fundamental components of number sense.*

Mental mathematics is a combination of cognitive strategies that enhance flexible thinking and number sense. It is calculating mentally without the use of external memory aids.

Mental mathematics enables students to determine answers without paper and pencil. It improves computational fluency by developing efficiency, accuracy and flexibility.

“Even more important than performing computational procedures or using calculators is the greater facility that students need—more than ever before—with estimation and mental math” (National Council of Teachers of Mathematics, May 2005).

Students proficient with mental mathematics “... become liberated from calculator dependence, build confidence in doing mathematics, become more flexible thinkers and are more able to use multiple approaches to problem solving” (Rubenstein, 2001, p. 442).

Mental mathematics “... provides the cornerstone for all estimation processes, offering a variety of alternative algorithms and nonstandard techniques for finding answers” (Hope, 1988, p. v).

Estimation is used for determining approximate values or quantities or for determining the reasonableness of calculated values. It often uses benchmarks or referents. Students need to know when to estimate, how to estimate and what strategy to use.

Estimation assists individuals in making mathematical judgements and in developing useful, efficient strategies for dealing with situations in daily life.

## Problem Solving [PS]

*Learning through problem solving should be the focus of mathematics at all grade levels.*

Learning through problem solving should be the focus of mathematics at all grade levels. When students encounter new situations and respond to questions of the type, “How would you know?” or “How could you ...?”, the problem-solving approach is being modelled. Students develop their own problem-solving strategies by listening to, discussing and trying different strategies.

A problem-solving activity requires students to determine a way to get from what is known to what is unknown. If students have already been given steps to solve the problem, it is not a problem, but practice. A true problem requires students to use prior learning in new ways and contexts. Problem solving requires and builds depth of conceptual understanding and student engagement.

Problem solving is a powerful teaching tool that fosters multiple, creative and innovative solutions. Creating an environment where students openly seek and engage in a variety of strategies for solving problems empowers students to explore alternatives and develops confident, cognitive mathematical risk takers.

**Reasoning [R]**

*Mathematical reasoning helps students think logically and make sense of mathematics.*

Mathematical reasoning helps students think logically and make sense of mathematics. Students need to develop confidence in their abilities to reason and justify their mathematical thinking. High-order questions challenge students to think and develop a sense of wonder about mathematics.

Mathematical experiences in and out of the classroom provide opportunities for students to develop their ability to reason. Students can explore and record results, analyze observations, make and test generalizations from patterns, and reach new conclusions by building upon what is already known or assumed to be true.

Reasoning skills allow students to use a logical process to analyze a problem, reach a conclusion and justify or defend that conclusion.

**Technology [T]**

*Technology contributes to the learning of a wide range of mathematical outcomes and enables students to explore and create patterns, examine relationships, test conjectures and solve problems.*

Technology contributes to the learning of a wide range of mathematical outcomes and enables students to explore and create patterns, examine relationships, test conjectures and solve problems.

Technology can be used to:

- explore and demonstrate mathematical relationships and patterns
- organize and display data
- extrapolate and interpolate
- assist with calculation procedures as part of solving problems
- decrease the time spent on computations when other mathematical learning is the focus
- reinforce the learning of basic facts
- develop personal procedures for mathematical operations
- create geometric patterns
- simulate situations
- develop number sense.

Technology contributes to a learning environment in which the growing curiosity of students can lead to rich mathematical discoveries at all grade levels.

## Visualization [V]

*Visualization is fostered through the use of concrete materials, technology and a variety of visual representations.*

Visualization “involves thinking in pictures and images, and the ability to perceive, transform and recreate different aspects of the visual-spatial world” (Armstrong, 1993, p. 10). The use of visualization in the study of mathematics provides students with opportunities to understand mathematical concepts and make connections among them.

Visual images and visual reasoning are important components of number, spatial and measurement sense. Number visualization occurs when students create mental representations of numbers.

Being able to create, interpret and describe a visual representation is part of spatial sense and spatial reasoning. Spatial visualization and reasoning enable students to describe the relationships among and between 3-D objects and 2-D shapes.

Measurement visualization goes beyond the acquisition of specific measurement skills. Measurement sense includes the ability to determine when to measure, when to estimate and which estimation strategies to use (Shaw and Cliatt, 1989).

## Nature of Mathematics

- *Change*
- *Constancy*
- *Number Sense*
- *Patterns*
- *Relationships*
- *Spatial Sense*
- *Uncertainty*

Mathematics is one way of trying to understand, interpret and describe our world. There are a number of components that define the nature of mathematics and these are woven throughout this curriculum guide. The components are change, constancy, number sense, patterns, relationships, spatial sense and uncertainty.

## Change

*Change is an integral part of mathematics and the learning of mathematics.*

It is important for students to understand that mathematics is dynamic and not static. As a result, recognizing change is a key component in understanding and developing mathematics.

Within mathematics, students encounter conditions of change and are required to search for explanations of that change. To make predictions, students need to describe and quantify their observations, look for patterns, and describe those quantities that remain fixed and those that change. For example, the sequence 4, 6, 8, 10, 12, ... can be described as:

- the number of a specific colour of beads in each row of a beaded design
- skip counting by 2s, starting from 4
- an arithmetic sequence, with first term 4 and a common difference of 2
- a linear function with a discrete domain

(Steen, 1990, p. 184).

## Constancy

*Constancy is described by the terms stability, conservation, equilibrium, steady state and symmetry.*

Different aspects of constancy are described by the terms stability, conservation, equilibrium, steady state and symmetry (AAAS-Benchmarks, 1993, p.270). Many important properties in mathematics and science relate to properties that do not change when outside conditions change. Examples of constancy include the following:

- The ratio of the circumference of a teepee to its diameter is the same regardless of the length of the teepee poles.
- The sum of the interior angles of any triangle is  $180^\circ$ .
- The theoretical probability of flipping a coin and getting heads is 0.5.

Some problems in mathematics require students to focus on properties that remain constant. The recognition of constancy enables students to solve problems involving constant rates of change, lines with constant slope, direct variation situations or the angle sums of polygons.

## Number Sense

*An intuition about number is the most important foundation of a numerate child.*

Number sense, which can be thought of as intuition about numbers, is the most important foundation of numeracy (British Columbia Ministry of Education, 2000, p.146).

A true sense of number goes well beyond the skills of simply counting, memorizing facts and the situational rote use of algorithms. Mastery of number facts is expected to be attained by students as they develop their number sense. This mastery allows for facility with more complex computations but should not be attained at the expense of an understanding of number.

Number sense develops when students connect numbers to their own real-life experiences and when students use benchmarks and referents. This results in students who are computationally fluent and flexible with numbers and who have intuition about numbers. The evolving number sense typically comes as a by product of learning rather than through direct instruction. It can be developed by providing rich mathematical tasks that allow students to make connections to their own experiences and their previous learning.

## Patterns

*Mathematics is about recognizing, describing and working with numerical and non-numerical patterns.*

Mathematics is about recognizing, describing and working with numerical and non-numerical patterns. Patterns exist in all strands of mathematics.

Working with patterns enables students to make connections within and beyond mathematics. These skills contribute to students' interaction with, and understanding of, their environment.

Patterns may be represented in concrete, visual or symbolic form. Students should develop fluency in moving from one representation to another.

Students must learn to recognize, extend, create and use mathematical patterns. Patterns allow students to make predictions and justify their reasoning when solving routine and non-routine problems.

Learning to work with patterns in the early grades helps students develop algebraic thinking, which is foundational for working with more abstract mathematics.

## Relationships

*Mathematics is used to describe and explain relationships.*

Mathematics is one way to describe interconnectedness in a holistic worldview. Mathematics is used to describe and explain relationships. As part of the study of mathematics, students look for relationships among numbers, sets, shapes, objects and concepts. The search for possible relationships involves collecting and analyzing data and describing relationships visually, symbolically, orally or in written form.

## Spatial Sense

*Spatial sense offers a way to interpret and reflect on the physical environment.*

Spatial sense involves visualization, mental imagery and spatial reasoning. These skills are central to the understanding of mathematics.

Spatial sense is developed through a variety of experiences and interactions within the environment. The development of spatial sense enables students to solve problems involving 3-D objects and 2-D shapes and to interpret and reflect on the physical environment and its 3-D or 2-D representations.

Some problems involve attaching numerals and appropriate units (measurement) to dimensions of shapes and objects. Spatial sense allows students to make predictions about the results of changing these dimensions; e.g., doubling the length of the side of a square increases the area by a factor of four. Ultimately, spatial sense enables students to communicate about shapes and objects and to create their own representations.

**Uncertainty**

*Uncertainty is an inherent part of making predictions.*

In mathematics, interpretations of data and the predictions made from data may lack certainty.

Events and experiments generate statistical data that can be used to make predictions. It is important to recognize that these predictions (interpolations and extrapolations) are based upon patterns that have a degree of uncertainty.

The quality of the interpretation is directly related to the quality of the data. An awareness of uncertainty allows students to assess the reliability of data and data interpretation.

Chance addresses the predictability of the occurrence of an outcome. As students develop their understanding of probability, the language of mathematics becomes more specific and describes the degree of uncertainty more accurately.

**Essential Graduation Learnings**

Essential graduation learnings are statements describing the knowledge, skills and attitudes expected of all students who graduate from high school. Essential graduation learnings are cross-curricular in nature and comprise different areas of learning: *aesthetic expression, citizenship, communication, personal development, problem solving, technological competence and spiritual and moral development.*

**Aesthetic Expression**

*Graduates will be able to respond with critical awareness to various forms of the arts and be able to express themselves through the arts.*

**Citizenship**

*Graduates will be able to assess social, cultural, economic and environmental interdependence in a local and global context.*

**Communication**

*Graduates will be able to use the listening, viewing, speaking, reading and writing modes of language(s) and mathematical and scientific concepts and symbols to think, learn and communicate effectively.*

**Personal Development**

*Graduates will be able to continue to learn and to pursue an active, healthy lifestyle.*

**Problem Solving**

*Graduates will be able to use the strategies and processes needed to solve a wide variety of problems, including those requiring language and mathematical and scientific concepts.*



## Technological Competence

*Graduates will be able to use a variety of technologies, demonstrate an understanding of technological applications, and apply appropriate technologies for solving problems.*

## Spiritual and Moral Development

*Graduates will be able to demonstrate an understanding and appreciation for the place of belief systems in shaping the development of moral values and ethical conduct.*

See *Foundations for the Atlantic Canada Mathematics Curriculum*, pages 4-6.

The mathematics curriculum is designed to make a significant contribution towards students' meeting each of the essential graduation learnings (EGLs), with the communication, problem-solving and technological competence EGLs relating particularly well to the mathematical processes.

## Outcomes and Achievement Indicators

The curriculum is stated in terms of general outcomes, specific outcomes and achievement indicators.

### *General Outcomes*

**General outcomes** are overarching statements about what students are expected to learn in each course.

### *Specific Outcomes*

**Specific outcomes** are statements that identify the specific skills, understanding and knowledge that students are required to attain by the end of a given course.

In the specific outcomes, the word *including* indicates that any ensuing items must be addressed to fully meet the learning outcome. The phrase *such as* indicates that the ensuing items are provided for illustrative purposes or clarification, and are not requirements that must be addressed to fully meet the learning outcome.

### *Achievement Indicators*

**Achievement indicators** are samples of how students may demonstrate their achievement of the goals of a specific outcome. The range of samples provided is meant to reflect the scope of the specific outcome.

Specific curriculum outcomes represent the means by which students work toward accomplishing the general curriculum outcomes and ultimately, the essential graduation learnings.

## Program Organization

Program Level	Course 1	Course 2	Course 3	Course 4
Advanced	Mathematics 1201	Mathematics 2200	Mathematics 3200	Calculus 3208
Academic		Mathematics 2201	Mathematics 3201	
Applied	Mathematics 1202	Mathematics 2202	Mathematics 3202	

The applied program is designed to provide students with the mathematical understandings and critical-thinking skills identified for entry into the majority of trades and for direct entry into the workforce.

The academic and advanced programs are designed to provide students with the mathematical understandings and critical-thinking skills identified for entry into post-secondary programs. Students who complete the advanced program will be better prepared for programs that require the study of calculus.

The programs aim to prepare students to make connections between mathematics and its applications and to become numerate adults, using mathematics to contribute to society.

## Summary

The conceptual framework for Grades 10-12 Mathematics (p. 3) describes the nature of mathematics, mathematical processes and the mathematical concepts to be addressed. The components are not meant to stand alone. Activities that take place in the mathematics classroom should result from a problem-solving approach, be based on mathematical processes and lead students to an understanding of the nature of mathematics through specific knowledge, skills and attitudes among and between topics.

# ASSESSMENT AND EVALUATION

## Purposes of Assessment

What learning is assessed and evaluated, how it is assessed and evaluated, and how results are communicated send clear messages to students and others about what is really valued.

*Assessment* techniques are used to gather information for evaluation. Information gathered through assessment helps teachers determine students' strengths and needs in their achievement of mathematics and guides future instructional approaches.

Teachers are encouraged to be flexible in assessing the learning success of all students and to seek diverse ways in which students might demonstrate what they know and are able to do.

*Evaluation* involves the weighing of the assessment information against a standard in order to make an evaluation or judgment about student achievement.

Assessment has three interrelated purposes:

- assessment *for* learning to guide and inform instruction;
- assessment *as* learning to involve students in self-assessment and setting goals for their own learning; and
- assessment *of* learning to make judgements about student performance in relation to curriculum outcomes.

## Assessment for Learning

Assessment *for* learning involves frequent, interactive assessments designed to make student understanding visible. This enables teachers to identify learning needs and adjust teaching accordingly. It is an ongoing process of teaching and learning.

Assessment *for* learning:

- requires the collection of data from a range of assessments as investigative tools to find out as much as possible about what students know
- provides descriptive, specific and instructive feedback to students and parents regarding the next stage of learning
- actively engages students in their own learning as they assess themselves and understand how to improve performance.

**Assessment as Learning**

Assessment *as* learning actively involves students' reflection on their learning and monitoring of their own progress. It focuses on the role of the student as the critical connector between assessment and learning, thereby developing and supporting metacognition in students.

Assessment *as* learning:

- supports students in critically analysing their learning related to learning outcomes
- prompts students to consider how they can continue to improve their learning
- enables students to use information gathered to make adaptations to their learning processes and to develop new understandings.

**Assessment of Learning**

Assessment *of* learning involves strategies to confirm what students know, demonstrate whether or not they have met curriculum outcomes, or to certify proficiency and make decisions about students' future learning needs. Assessment *of* learning occurs at the end of a learning experience that contributes directly to reported results.

Traditionally, teachers relied on this type of assessment to make judgments about student performance by measuring learning after the fact and then reporting it to others. Used in conjunction with the other assessment processes previously outlined, however, assessment *of* learning is strengthened.

Assessment *of* learning:

- provides opportunities to report evidence to date of student achievement in relation to learning outcomes, to parents/guardians and other stakeholders
- confirms what students know and can do
- occurs at the end of a learning experience using a variety of tools.

Because the consequences of assessment *of* learning are often far-reaching, teachers have the responsibility of reporting student learning accurately and fairly, based on evidence obtained from a variety of contexts and applications.

## **Assessment Strategies**

Assessment techniques should match the style of learning and instruction employed. Several options are suggested in this curriculum guide from which teachers may choose, depending on the curriculum outcomes, the class and school/district policies.

### **Observation (formal or informal)**

This technique provides a way of gathering information fairly quickly while a lesson is in progress. When used formally, the student(s) would be aware of the observation and the criteria being assessed. Informally, it could be a frequent, but brief, check on a given criterion. Observation may offer information about the participation level of a student for a given task, use of a concrete model or application of a given process. The results may be recorded in the form of checklists, rating scales or brief written notes. It is important to plan in order that specific criteria are identified, suitable recording forms are ready, and all students are observed within a reasonable period of time.

### **Performance**

This curriculum encourages learning through active participation. Many of the curriculum outcomes promote skills and their applications. In order for students to appreciate the importance of skill development, it is important that assessment provide feedback on the various skills. These may be the correct manner in which to use a manipulative, the ability to interpret and follow instructions, or to research, organize and present information. Assessing performance is most often achieved through observing the process.

### **Paper and Pencil**

These techniques can be formative or summative. Whether as part of learning, or a final statement, students should know the expectations for the exercise and how it will be assessed. Written assignments and tests can be used to assess knowledge, understanding and application of concepts. They are less successful at assessing processes and attitudes. The purpose of the assessment should determine what form of paper and pencil exercise is used.

### **Journal**

Journals provide an opportunity for students to express thoughts and ideas in a reflective way. By recording feelings, perceptions of success, and responses to new concepts, a student may be helped to identify his or her most effective learning style. Knowing how to learn in an effective way is powerful information. Journal entries also give indicators of developing attitudes to mathematical concepts, processes and skills, and how these may be applied in the context of society. Self-assessment, through a journal, permits a student to consider strengths and weaknesses, attitudes, interests and new ideas. Developing patterns may help in career decisions and choices of further study.

**Interview**

This curriculum promotes understanding and applying mathematics concepts. Interviewing a student allows the teacher to confirm that learning has taken place beyond simple factual recall. Discussion allows a student to display an ability to use information and clarify understanding. Interviews may be a brief discussion between teacher and student or they may be more extensive. Such conferences allow students to be proactive in displaying understanding. It is helpful for students to know which criteria will be used to assess formal interviews. This assessment technique provides an opportunity to students whose verbal presentation skills are stronger than their written skills.

**Presentation**

The curriculum includes outcomes that require students to analyze and interpret information, to be able to work in teams, and to communicate information. These activities are best displayed and assessed through presentations. These can be given orally, in written/pictorial form, by project summary, or by using electronic systems such as video or computer software. Whatever the level of complexity, or format used, it is important to consider the curriculum outcomes as a guide to assessing the presentation. The outcomes indicate the process, concepts and context for which a presentation is made.

**Portfolio**

Portfolios offer another option for assessing student progress in meeting curriculum outcomes over a more extended period of time. This form of assessment allows the student to be central to the process. There are decisions about the portfolio, and its contents, which can be made by the student. What is placed in the portfolio, the criteria for selection, how the portfolio is used, how and where it is stored, and how it is evaluated are some of the questions to consider when planning to collect and display student work in this way. The portfolio should provide a long-term record of growth in learning and skills. This record of growth is important for individual reflection and self-assessment, but it is also important to share with others. For all students, it is exciting to review a portfolio and see the record of development over time.

# INSTRUCTIONAL FOCUS

## Planning for Instruction

Consider the following when planning for instruction:

- Integration of the mathematical processes within each topic is expected.
- By decreasing emphasis on rote calculation, drill and practice, and the size of numbers used in paper and pencil calculations, more time is available for concept development.
- Problem solving, reasoning and connections are vital to increasing mathematical fluency and must be integrated throughout the program.
- There should be a balance among mental mathematics and estimation, paper and pencil exercises, and the use of technology, including calculators and computers. Concepts should be introduced using manipulatives and be developed concretely, pictorially and symbolically.
- Students bring a diversity of learning styles and cultural backgrounds to the classroom. They will be at varying developmental stages.

## Teaching Sequence

The curriculum guide for Calculus 3208 is organized by units. This is only a suggested teaching order for the course. There are a number of combinations of sequences that would be appropriate.

Each two page spread lists the topic, general outcome, and specific outcome.

## Instruction Time Per Unit

The suggested number of hours of instruction per unit is listed in the guide at the beginning of each unit. The number of suggested hours includes time for completing assessment activities, reviewing and evaluating. The timelines at the beginning of each unit are provided to assist in planning. The use of these timelines is not mandatory. However, it is mandatory that all outcomes are taught during the school year, so a long term plan is advised. Teaching of the outcomes is ongoing, and may be revisited as necessary.

## Resources

The authorized resources for Newfoundland and Labrador students and teachers are *Pre-Calculus 12* (McGraw-Hill Ryerson) and *Single Variable Essential Calculus Second Edition* (Nelson). Column four of the curriculum guide references these texts for this reason.

Teachers may use any other resource, or combination of resources, to meet the required specific outcomes.

## GENERAL AND SPECIFIC OUTCOMES

### GENERAL AND SPECIFIC OUTCOMES WITH ACHIEVEMENT INDICATORS (pages 19-234)

This section presents general and specific outcomes with corresponding achievement indicators and is organized by unit. The list of indicators contained in this section is not intended to be exhaustive but rather to provide teachers with examples of evidence of understanding that may be used to determine whether or not students have achieved a given specific outcome. Teachers may use any number of these indicators or choose to use other indicators as evidence that the desired learning has been achieved. Achievement indicators should also help teachers form a clear picture of the intent and scope of each specific outcome.

Calculus 3208 is organized into eight units: *Pre-Calculus, Limits and Continuity, Rational Functions, Derivatives, Applications of Derivatives, Calculus of Trigonometry, Calculus of Exponential, Logarithmic, and Hyperbolic Trigonometric Functions, and Antidifferentiation and Integration.*



# Pre-Calculus

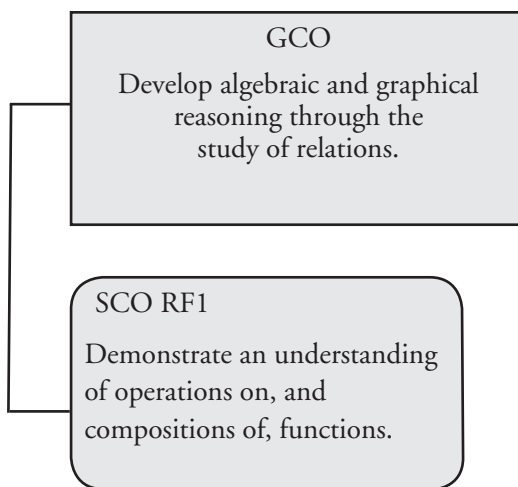
Suggested Time: 5 Hours

# Unit Overview

## Focus and Context

In this unit, students will create new functions by adding, subtracting, multiplying, dividing, and composing two or more given functions. They will sketch the graph of the function and determine its domain and range.

## Outcomes Framework



## SCO Continuum

Mathematics 2200	Mathematics 3200	Calculus 3208
Relations and Functions		
RF2 Graph and analyze absolute value functions (limited to linear and quadratic functions) to solve problems. [C, PS, R, T, V]	not addressed	RF1 Demonstrate an understanding of operations on, and compositions of, functions. [CN, R, T, V]
RF11 Graph and analyze reciprocal functions (limited to the reciprocal of linear and quadratic functions). [CN, R, T, V]		

## Mathematical Processes

[C] Communication	[PS] Problem Solving
[CN] Connections	[R] Reasoning
[ME] Mental Mathematics and Estimation	[T] Technology
	[V] Visualization

## Relations and Functions

### Outcomes

*Students will be expected to*

**RF1 Demonstrate an understanding of operations on, and compositions of, functions.**

[CN, R, T, V]

### Achievement Indicators:

RF1.1 Write a function  $h(x)$  as the sum, difference, product or quotient of two or more functions.

RF1.2 Sketch the graph of a function that is the sum, difference, product or quotient of two functions, given their graphs.

RF1.3 Determine the domain and range of a function that is the sum, difference, product or quotient of two functions.

### Elaborations – Strategies for Learning and Teaching

Students should work with the operations of functions. They add, subtract, multiply, and divide functions to create new functions. Students should then progress to composite functions where they determine functional values and identify the domain and range of the composite function. They are not expected to solve contextual problems.

In Mathematics 1201, students were introduced to function notation (RF8). A review of function notation is warranted to prepare students for operations on functions which require them to read functional values from graphs. In Mathematics 2200, students were exposed to absolute value graphs limited to linear or quadratic functions (RF2). They also determined restrictions on values for the variable in a radical equation (AN3). Students should continue to identify restrictions as they determine the domain of a combined function involving a square root.

Using the equations  $f(x)$  and  $g(x)$ , students should write the equation of a new function  $h(x)$  that will represent the sum, difference, product, and quotient. They should be exposed to a linear function, quadratic function, square root function where the radicand contains variables that are first degree, absolute value of a linear function and a rational function.

$$h(x) = f(x) + g(x) = (f + g)(x)$$

$$h(x) = f(x) - g(x) = (f - g)(x)$$

$$h(x) = f(x) \times g(x) = (f \times g)(x)$$

$$h(x) = \frac{f(x)}{g(x)} = \left(\frac{f}{g}\right)(x), \quad g(x) \neq 0$$

When writing the equation of the quotient of  $f(x)$  and  $g(x)$ , students may have to be reminded to factor the expression in order to simplify the quotient  $\left(\frac{f}{g}\right)(x)$  and check for non-permissible values. Students are aware from Mathematics 2200 that vertical asymptotes occur where the denominator is zero (RF11). Ask them to simplify a rational expression such as  $h(x) = \frac{3x+12}{x^2-x-20}$ . Students may initially think the vertical asymptotes occur at  $x = -4$  and  $x = 5$ . Once they factor the numerator and denominator, however, they should notice that there is a common factor  $x + 4$ . This implies there is a point of discontinuity at  $x = -4$ . Students should graph the function and analyze the table of values to verify a hole exists at  $x = -4$  and a vertical asymptote occurs at  $x = 5$ . Points of discontinuity will be explored in greater detail when students are introduced to limits.

It is not the intent for students to algebraically determine the points of discontinuity at this time. It is important, however, to discuss how non-permissible values can manifest themselves on the graph of a rational function.

## General Outcome: Develop algebraic and graphical reasoning through the study of relations.

### Suggested Assessment Strategies

#### Performance

- Ask students to use graphing technology to investigate the sum or difference of a linear and a quadratic function. They should identify the domain and range of the new function.

(RF1.1, RF1.2, RF1.3)

#### Journal

- Ask students to explain why the function  $f(x) = \frac{2}{x^2}$  does not have a domain of all real numbers.

(RF1.3)

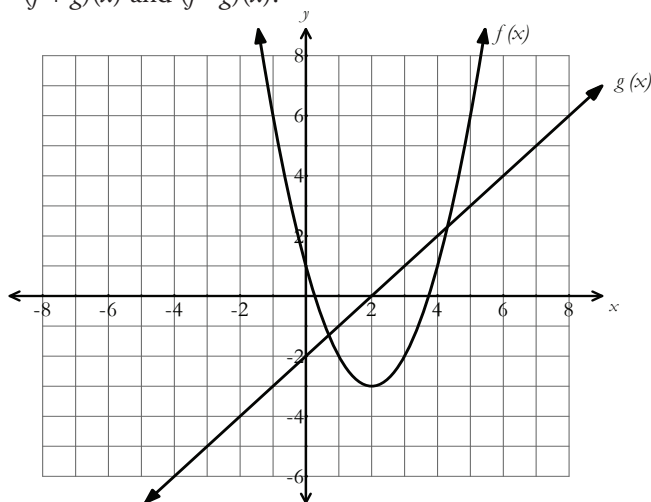
#### Interview

- Ask students to respond to the following:
  - If  $f(x)$  is a linear function and  $g(x)$  is a quadratic function, discuss the type of graph produced by  $(f + g)(x)$ .
  - If both  $f(x)$  and  $g(x)$  are linear functions, discuss the type of graph produced by  $(f - g)(x)$ .
  - If both  $f(x)$  and  $g(x)$  are linear functions, discuss the type of graph produced by  $(f \times g)(x)$ .

(RF1.1, RF1.2)

#### Paper and Pencil

- Given  $f(x)$  and  $g(x)$  as shown, ask students to create the graphs of  $(f + g)(x)$  and  $(f - g)(x)$ .



(RF1.2)

- Ask students to determine  $k(x) = \frac{f(x)}{g(x)}$  where  $g(x) = x + 1$  and  $f(x) = \sqrt{x + 2}$  and state the domain of  $k(x)$ .

(RF1.1, RF1.3)

### Resources/Notes

#### Authorized Resource

*Single Variable Essential Calculus  
Second Edition*

#### 1.2: A Catalog of Essential Functions

Student Book (SB): pp. 18 - 24  
Instructor's Guide (IG): pp. 7 - 18  
Power Point Lectures (PPL):  
SteEC\_01\_02.ppt

#### Supplementary Resource

*Pre-Calculus 12*

#### 10.1: Sums and Differences of Functions

SB: pp. 474 - 487  
TR: pp. 258 - 262

#### 10.2: Products and Quotients of Functions

SB: pp. 488 - 498  
TR: pp. 263 - 267

#### Note:

Students are not responsible for combining functions that involve exponential functions and trigonometric functions in the *Pre-Calculus 12* resource.

**Relations and Functions**

**Outcomes**

*Students will be expected to*

**RF1 Continued...**

**Achievement Indicators:**

RF1.1, RF1.2, RF1.3

*Continued*

**Elaborations – Strategies for Learning and Teaching**

Ask students how they can use the graph of  $f(x)$  and  $g(x)$  to draw the graph of the function  $h(x)$ . Encourage them to think about the  $y$ -coordinates of the points on the graphs of  $f(x)$  and  $g(x)$ . If students are asked to determine the sum of functions from a graph, for example, they have to add the corresponding  $y$ -values for each value of  $x$ . It may be beneficial for students to organize the information in a table, choosing at least five values for  $x$ .

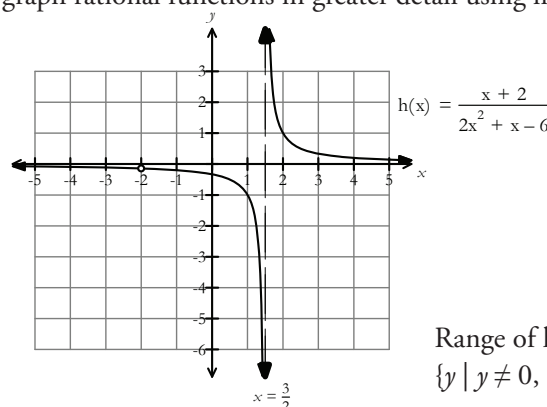
$x$	$f(x)$	$g(x)$	$h(x) = f(x) + g(x)$

As students form new functions by performing operations with functions, domain and range should be discussed. Students should reflect on the following functions that are familiar to them:

- for the domain of square root functions, the value of the radical must be greater than or equal to zero.
- for the domain of the quotient of functions, it is important to consider the possibility of division by zero.

Ask students to determine the equation of the function such as  $h(x) = \left(\frac{f}{g}\right)(x)$ , where  $f(x) = x + 2$  and  $g(x) = 2x^2 + x - 6$  and state the domain of  $h(x)$ . Students should conclude that since  $f(x)$  is linear and  $g(x)$  is quadratic, they have domains which include all real numbers. The domain of  $h(x)$  consists of all values that are in both the domain of  $f(x)$  and the domain of  $g(x)$ , excluding values where  $g(x) = 0$ . Ask students why there is a point of discontinuity at  $x = -2$  and a vertical asymptote at  $x = \frac{3}{2}$ .

Ask students to then determine the range of  $h(x) = \frac{x+2}{2x^2+x-6}$ . Using the graph as a visual aid often helps students identify the range. Students can either use graphing technology or use the graphs of  $f(x)$  and  $g(x)$  to produce the graph of its quotient. Later in this course, students will graph rational functions in greater detail using limits and derivatives.



Range of  $h(x)$ :  
 $\{y \mid y \neq 0, y \neq -\frac{1}{7}, y \in \mathbb{R}\}$

## General Outcome: Develop algebraic and graphical reasoning through the study of relations.

### Suggested Assessment Strategies

#### *Paper and Pencil*

- Provide students with several functions similar to the ones shown below.

$$\begin{aligned} \text{(i)} \quad & f(x) = |x - 2|, g(x) = 3, h(x) = (f + g)(x) \\ \text{(ii)} \quad & f(x) = 2x^2 + 5, g(x) = 4x - 1, h(x) = (f - g)(x) \\ \text{(iii)} \quad & f(x) = x + 5, g(x) = 4x - 1, h(x) = (f \times g)(x) \\ \text{(iv)} \quad & f(x) = x^2 + x - 6, g(x) = x - 2, h(x) = \left(\frac{f}{g}\right)(x) \end{aligned}$$

In groups of four, assign each student a specific task.

Student A will determine the combined function  $h(x)$ .

Student B will sketch the graph of  $h(x)$  and verify with technology.

Student C will determine the domain of  $h(x)$ .

Student D will determine the range of  $h(x)$ .

They should discuss their answers with each other and help identify errors if they occur.

(RF1.1, RF1.2, RF1.3)

- Ask students to copy and complete the following table and then graph  $h(x)$ . They should determine the domain and range.

$x$	$f(x) = -2x + 3$	$g(x) = -x^2 + 2$	$h(x) = (f + g)(x)$
-3			
-2			
-1			
0			
1			
2			
3			

(RF1.1, RF1.2, RF1.3)

- Ask students to create functions  $f(x)$  and  $g(x)$  such that the domain of the combined function  $h(x) = (f \times g)(x)$  is  $\{x \mid x \neq -1, 4, x \in \mathbb{R}\}$ .

(RF1.3)

- Given  $f(x) = 3x^2 - 4$ ,  $g(x) = 2x + 1$  and  $h(x) = \frac{1}{x}$ , ask students to determine each combined function, in simplest form.

$$\begin{aligned} \text{(i)} \quad & (f + g)(x) & \text{(ii)} \quad & (f \times h)(x) & \text{(iii)} \quad & (g - h)(x) \\ \text{(iv)} \quad & \frac{g(x)}{f(x)} & \text{(v)} \quad & f(x) g(x) h(x) & \text{(vi)} \quad & \frac{f(x)}{h(x)} \end{aligned}$$

(RF1.1)

### Resources/Notes

#### Authorized Resource

*Single Variable Essential Calculus  
Second Edition*

#### 1.2: A Catalog of Essential Functions

SB: pp. 18 - 24

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PPL: SteEC\_01\_02.ppt

#### Supplementary Resource

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TR: pp. 263 - 267

## Relations and Functions

### Outcomes

*Students will be expected to*

RF1 Continued...

#### Achievement Indicators:

RF1.4 *Determine the value of the composition of functions when evaluated at a point, including:*

- $f(f(a))$
- $f(g(a))$
- $g(f(a))$ .

RF1.5 *Determine the equation of the composite function given the equations of two functions  $f(x)$  and  $g(x)$ :*

- $f(f(x))$
- $f(g(x))$
- $g(f(x))$ .

RF1.6 *Sketch the graph of the composite function given the equations of two functions  $f(x)$  and  $g(x)$  and determine the domain and range.*

RF1.7 *Determine the original functions from a composition.*

### Elaborations – Strategies for Learning and Teaching

The knowledge of composition of functions is required later in this course when students are introduced to the differentiation rules, especially the chain rule. They need to have an understanding of the order of composition (i.e., which is the "inner" and which is the "outer" function in the composition). Students should work through examples where the functions are linear, quadratic, square root, rational and absolute value.

Special attention needs to be given to the notation of composition, either written as  $(f \circ g)(x)$  or  $(f(g(x)))$ , and that composition should not be confused with multiplication, written  $f(x) \times g(x)$ . Students should explore various orders of composition, such as  $f(f(x))$ ,  $g(f(x))$  and  $f(g(x))$ . With reference to  $f(g(x))$ , for example, discuss with students that the output of  $g(x)$  becomes the input of  $f(x)$ . Ask students to find both a numerical solution if a specific domain value is used (i.e.,  $f(g(3))$ ) as well as a new function,  $h(x)$ , if given the equations of two functions.

Ask students to determine the composite function  $h(x)$  and then sketch its graph using technology or a table of values. This visual will help students when they determine the domain and range.

When considering the domain, it is important for students to find the domain of the composite function in terms of the inner function. That is, the domain of  $(f(g(x)))$  is the set of elements  $x$  in the domain of  $g$  such that  $g(x)$  is in the domain of  $f$ . Ask students to determine the domain of a composite function such as  $h(x) = (f(g(x)))$  where  $g(x) = x + 2$  and  $f(x) = \frac{1}{x}$ . Some students may incorrectly write the following solution:

- The domain of  $g(x)$  is all real numbers and the domain of  $f(x)$  is all real numbers where  $x \neq 0$ . Combining these restrictions will result in the domain of the composition, all real numbers where  $x \neq 0$ .

Ask students to identify the error and to write the correct solution. They should conclude that it is necessary to take the restrictions on the inner function as well as the composite function. There is no restriction on the domain of  $g(x)$ . The restriction on the domain of  $h(x) = \frac{1}{x+2}$  is  $\{x \mid x \neq -2, x \in \mathbb{R}\}$ .

Given a composite function,  $h(x)$ , students should determine the possible functions for  $f(x)$  and  $g(x)$ . It is worth noting that there may be several possible solutions to these questions. Given the composition  $f(g(x)) = (x + 5)^2$ , for example, students may write the following two functions:  $f(x) = x^2$  and  $g(x) = x + 5$  or  $f(x) = (x + 2)^2$  and  $g(x) = x + 3$ .

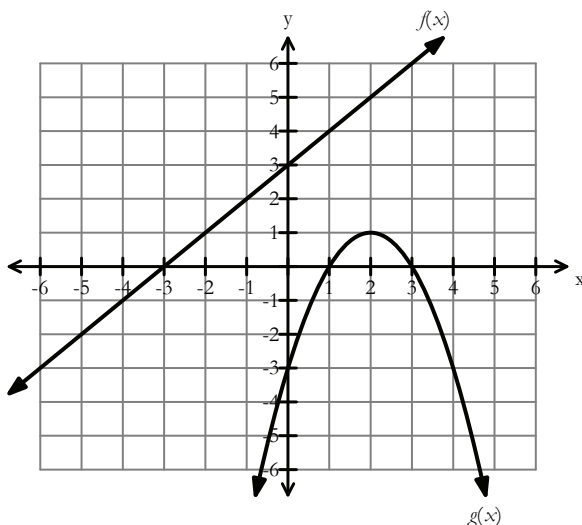


**General Outcome: Develop algebraic and graphical reasoning through the study of relations.**

### Suggested Assessment Strategies

#### *Paper and Pencil*

- Ask students to use the graphs of  $f(x)$  and  $g(x)$  to evaluate the following:



- $f(g(1))$
- $f(g(2))$
- $g(f(1))$
- $g(f(-3))$

(RF1.4)

- Given  $f(x) = \sqrt{x-2}$  and  $g(x) = x^2$ , ask students to sketch the graph of the composite function  $g(f(x))$ . They should determine the domain and range of  $f(x)$ ,  $g(x)$  and  $g(f(x))$ .

(RF1.6)

- Ask students to answer the following:

- Given  $h(x) = (x+3)^2$ , find two functions  $f(x)$  and  $g(x)$ , such that  $h(x) = g(f(x))$ .
- Given  $h(x) = \frac{1}{x-4}$ , find two functions  $f(x)$  and  $g(x)$ , such that  $h(x) = f(g(x))$ .

(RF1.7)

### Resources/Notes

#### Authorized Resource

*Single Variable Essential Calculus  
Second Edition*

#### 1.2: A Catalog of Essential Functions

SB: pp. 18 - 24

IG: pp. 7 - 18

PPL: SteEC\_01\_02.ppt

#### Supplementary Resource

*Pre-Calculus 12*

#### 10.3: Composite Functions

SB: pp. 499-509

TR: pp. 268 - 272



# Limits and Continuity

Suggested Time: 20 Hours

# Unit Overview

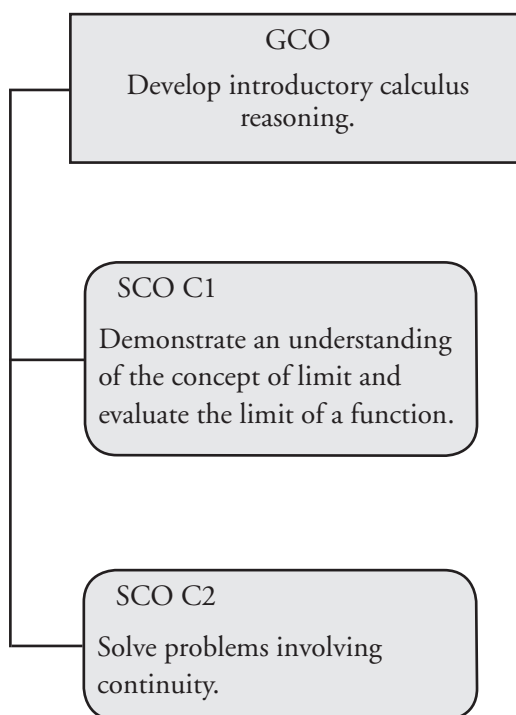
## Focus and Context

This unit presents concepts and terms that are fundamental to an understanding of calculus. Students explore the behaviour of functions as they approach certain  $x$ -values. They describe this behaviour of a function using limits. Later in this course, the concept of limit will become the basis for the development of the derivative.

Students evaluate the limiting behaviour of a function graphically, numerically and algebraically. They evaluate the limit of a polynomial function, a square root function, a rational function, and a function that contains a square root in the numerator or denominator.

Closely connected to the concept of a limit is continuity. Students analyze graphs and observe what makes them discontinuous. They work with the definition of continuity to determine if a function is continuous at a point.

## Outcomes Framework



## SCO Continuum

Mathematics 2200	Mathematics 3200	Calculus 3208
Relations and Functions	Relations and Functions	Calculus
<p>RF2 Graph and analyze absolute value functions (limited to linear and quadratic functions) to solve problems. [C, PS, R, T, V]</p> <p>RF10 Analyze geometric sequences and series to solve problems. [PS, R]</p> <p>RF11 Graph and analyze reciprocal functions (limited to the reciprocal of linear and quadratic functions). [CN, R T, V]</p>	<p>RF10 Demonstrate an understanding of factoring polynomials of degree greater than 2 (limited to polynomials of degree <math>\leq 5</math> with integral coefficients). [C, CN, ME]</p> <p>RF12 Graph and analyze radical functions (limited to functions involving one radical). [CN, R, T, V]</p>	<p>C1 Demonstrate an understanding of the concept of limit and evaluate the limit of a function. [C, CN, R, T, V]</p> <p>C2 Solve problems involving continuity. [C, CN, R, T, V]</p>
Algebra and Number		
<p>AN2 Solve problems that involve operations on radicals and radical expressions with numerical and variable radicands. [CN, ME, PS, R]</p>		

## Mathematical Processes

[C] Communication	[PS] Problem Solving
[CN] Connections	[R] Reasoning
[ME] Mental Mathematics and Estimation	[T] Technology
	[V] Visualization

## Calculus Reasoning

### Outcomes

*Students will be expected to*

**C1 Demonstrate an understanding of the concept of limit and evaluate the limit of a function.**

[C, CN, R, T, V]

### Achievement Indicator:

C1.1 *Using informal methods, explore the concept of a limit including one sided limits.*

### Elaborations – Strategies for Learning and Teaching

In Mathematics 2200, students determined the sum of an infinite geometric series (RF10). If the sequence of partial sums approached a definite value, they recognized the series as convergent. In the case of divergence, the partial sum fails to approach any finite value. It oscillates or grows without bound. Although students analyzed partial sums, the term *limit* was not introduced.

In this unit, students are introduced to the formal definition of a limit and methods for evaluating the limit of a polynomial function, a square root function, a rational function, and a function that contains a square root in the numerator or denominator. Direct substitution will sometimes result in a limit of an indeterminate form.

Students will solve problems involving the limits of trigonometric functions, exponential functions and logarithmic functions later in this course.

Teachers should use informal methods when introducing limits. Students will be exposed to limit notation and how to determine the value of a limit algebraically later in this unit.

The informal analysis of a limit should include reference to the idea of getting closer and closer to a value or a location. Ask students to add the terms of an infinite geometric series such as  $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$ . As they determine the sum, they need to reflect on how many terms they should use. As they create sums of a finite number of terms, called partial sums, they can determine if the sequence of partial sums gets closer and closer to a number.

Some students may refer to Google Earth. They may choose a general location such as Newfoundland and Labrador and zoom in closer and closer to target a specific location. Ask students to reflect on these responses and how they support the concept of a limit.

Limits should also be explored using a table of values. Using a function such as  $f(x) = 3x - 1$ , ask students to determine the behaviour of  $f(x)$  as  $x$  approaches 2. They should construct a table that shows values of  $f(x)$  for two sets of  $x$  values; one set that approaches 2 from the left and one that approaches 2 from the right.

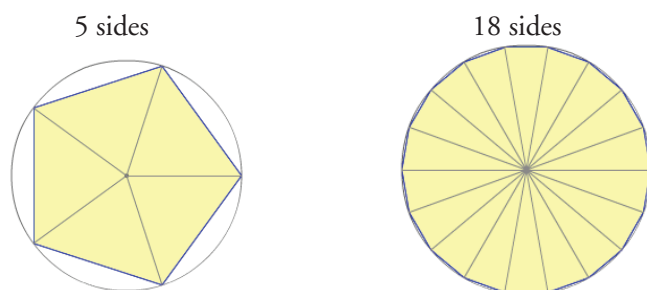
$x$	1.9	1.99	1.999	2.0	2.001	2.01	2.1
$f(x)$	4.7	4.97	4.997	?	5.003	5.03	5.3

Students should conclude that the closer  $x$  is to 2, the closer  $f(x)$  is to 5. Ask them to graph the function to further support their estimate of the limit being 5.

**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Observation*

- To introduce and discuss the concept of a limit, the following activity could be explored.

Ask students to use an applet to visualize how the area of a polygon becomes a closer and closer approximation to the area of a circle as the number of sides in the polygon increases. They should start with a three sided regular polygon and increase the number of sides.



(C1.1)

**Resources/Notes****Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**1.3: The Limit of a Function**

Student Book (SB): pp. 24 - 35

Instructor's Guide (IG): pp. 19 - 34

Power Point Lectures (PPL):

SteEC\_01\_03.ppt

Video Examples: 1

**Suggested Resource****Resource Link:**

[www.k12pl.nl.ca/curr/10-12/math/3208/res-link/limit.html](http://www.k12pl.nl.ca/curr/10-12/math/3208/res-link/limit.html)

- applet to demonstrate that as the number of sides in a polygon increases, its area becomes an approximation of the area of the circle

## Calculus Reasoning

### Outcomes

Students will be expected to

C1 Continued...

#### Achievement Indicators:

C1.1 Continued

C1.2 Using informal methods, establish that the limit of  $\frac{1}{x}$  as  $x$  approaches infinity is zero.

### Elaborations – Strategies for Learning and Teaching

Similarly, ask students to create a table of values to determine the behaviour of a rational function such as  $f(x) = \frac{x^2+2x-3}{x-1}$  as  $x$  approaches 1. They should consider the restrictions on the domain of  $f(x)$ .

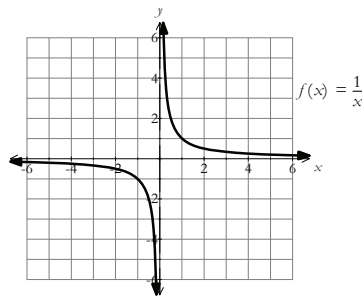
$x$	0.9	0.99	0.999	1.0	1.001	1.01	1.1
$f(x)$	3.9	3.99	3.999	?	4.001	4.01	4.1

Using the information from the table, students should make the following conclusions:

- As  $x$  gets closer to 1 from the left and the right,  $f(x)$  is getting closer to 4 (i.e., the limit is 4).
- $f(x)$  has a limit as  $x$  approaches 1 even though the function is not defined at  $x = 1$ .
- When evaluating a limit, what is happening at the point in question is not important. It is about what the graph is doing around the point.

For the function  $f(x) = \frac{1}{x}$ , provide students a table with the indicated  $x$ -values. Ask students to determine the functional values, and produce its graph. They should continue to determine the restrictions on the domain of  $f(x)$  and conclude that  $x \neq 0$ .

$x$	-5	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	1	5
$f(x)$	$-\frac{1}{5}$	-1	-2	-4	?	4	2	1	$\frac{1}{5}$



Using the table and graph as a guide, ask students to reflect on the following questions:

- What value is  $f(x)$  approaching as  $x$  becomes a larger positive number?
- What value is  $f(x)$  approaching as  $x$  becomes a larger negative number?
- Will the value of  $f(x)$  ever equal zero? Explain your reasoning.

Discussions around the above questions should include the word “limit”. Using the visual representation, students should notice that as  $x$  approaches infinity, the value of  $f(x)$  approaches zero.



**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Paper and Pencil*

- Ask students to use a table of values to estimate the limit of the function as  $x$  approaches a specific value. This could be used as an exit card.

(i)  $y = (x + 3)^2$  as  $x$  approaches -1

(ii)  $f(x) = \frac{2}{x+5}$  as  $x$  approaches -5

(C1.1)

- Using graphing technology, ask students to investigate the value of the limit as  $x$  approaches infinity for the following:

(i)  $f(x) = \frac{2}{x}$

(ii)  $f(x) = \frac{10}{x}$

(iii)  $f(x) = \frac{100}{x}$

(C1.2)

**Resources/Notes****Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**1.3: The Limit of a Function**

SB: pp. 24 - 35

IG: pp. 19 - 34

PPL: SteEC\_01\_03.ppt

## Calculus Reasoning

### Outcomes

Students will be expected to

C1 Continued...

#### Achievement Indicators:

C1.3 Explore the concept of limit and the notation used in expressing the limit of a function:

- $\lim_{x \rightarrow a^+} f(x)$
- $\lim_{x \rightarrow a^-} f(x)$
- $\lim_{x \rightarrow a} f(x)$ .

C1.4 Determine the value of the limit of a function as the variable approaches a real number

- by using a provided graph, including piecewise functions
- by using a table of values.

C1.5 Evaluate one-sided limits using a graph.

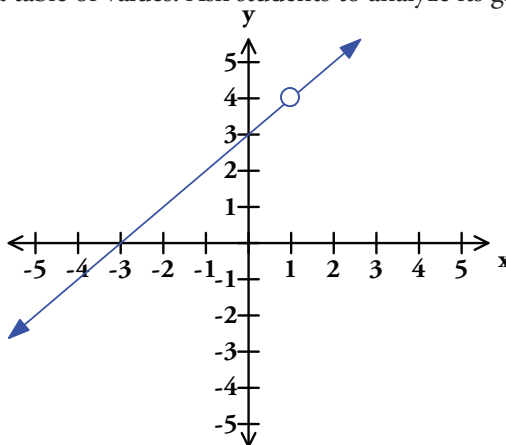
### Elaborations – Strategies for Learning and Teaching

Up to this point, the concept of limit has been discussed using the words "approaches" or "getting close to". Students should now be formally introduced to the definition of a limit,  $\lim_{x \rightarrow a} f(x) = L$ , and its notation.

With reference to the previous graph of  $f(x) = \frac{1}{x}$ , ask students to use words to express the limit notation of a function.

Notation	Read as
$\lim_{x \rightarrow \infty} f(x) = 0$	"the limit of $f(x)$ as $x$ approaches infinity is zero"
$\lim_{x \rightarrow -\infty} f(x) = 0$	"the limit of $f(x)$ as $x$ approaches negative infinity is zero"
$\lim_{x \rightarrow 5^+} f(x) = \frac{1}{5}$	"the limit of $f(x)$ as $x$ approaches 5 from the right is $\frac{1}{5}$ "
$\lim_{x \rightarrow 5^-} f(x) = \frac{1}{5}$	"the limit of $f(x)$ as $x$ approaches 5 from the left is $\frac{1}{5}$ "
$\lim_{x \rightarrow 5} f(x) = \frac{1}{5}$	"the limit of $f(x)$ as $x$ approaches 5 is $\frac{1}{5}$ "

Remind students when they are determining limits, they should continue to think about what  $y$ -value the graph is approaching as  $x$  approaches  $a$ . Students previously explored the function  $f(x) = \frac{x^2+2x-3}{x-1}$  using a table of values. Ask students to analyze its graph as  $x$  approaches 1.



Students should notice the graph of  $f(x)$  has a hole, or point of discontinuity, at  $x = 1$ . A point of discontinuity is a part of the graph that is undefined at a particular point, but there is no asymptote at that point. Students will determine points of discontinuity when they work with rational functions later in this course.

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**General Outcome: Develop introductory calculus reasoning.**

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**Suggested Assessment Strategies***Journal*

- Ask students to explain in their own words what is meant by the equation  $\lim_{x \rightarrow 3} f(x) = 7$ .

They should explain if it possible for this statement to be true and yet  $f(3) = 10$ .

(C1.3)

**Resources/Notes****Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**1.3: The Limit of a Function**

SB: pp. 24 - 35

IG: pp. 19 - 34

PPL: SteEC\_01\_03.ppt

Video Examples: 6 and 7

## Calculus Reasoning

## Outcomes

Students will be expected to

C1 Continued...

## Achievement Indicators:

C1.4, C1.5 *Continued*

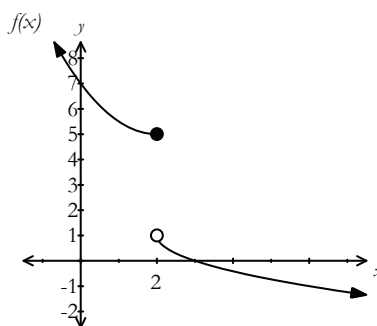
## Elaborations—Strategies for Learning and Teaching

Remind students that the function does not actually have to exist at a certain point for the limit to exist. In this case, as  $x$  moves towards 1 (from both sides), the function is approaching  $y = 4$ . Students should conclude that the limit is 4. This can then be written using limit notation:

$$\begin{array}{ccc} \lim_{x \rightarrow 1^+} f(x) = 4 & \Longrightarrow & \lim_{x \rightarrow 1} f(x) = 4 \\ \lim_{x \rightarrow 1^-} f(x) = 4 & & \end{array}$$

It is important for teachers to emphasize that the left and right hand limits must be equal in order for a limit to exist.

In Mathematics 2200, students wrote the absolute value of linear and quadratic functions as a piecewise function (RF2). They should now determine the value of the limit of a function as the variable approaches a real number in a piecewise function. Using the graph, for example, ask questions to help students determine the limit of  $f(x)$  as  $x$  approaches 2.

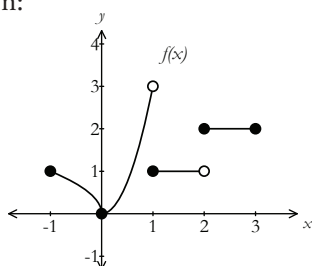


- What is the limiting value that  $f(x)$  approaches as  $x$  approaches 2 from the left?
- What is the limiting value that  $f(x)$  approaches as  $x$  approaches 2 from the right?
- How can these be written using limit notation?

Students should observe that, since  $\lim_{x \rightarrow 2^-} f(x) = 5$  and  $\lim_{x \rightarrow 2^+} f(x) = 1$ , the  $\lim_{x \rightarrow 2} f(x)$  does not exist.

**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Paper and Pencil*

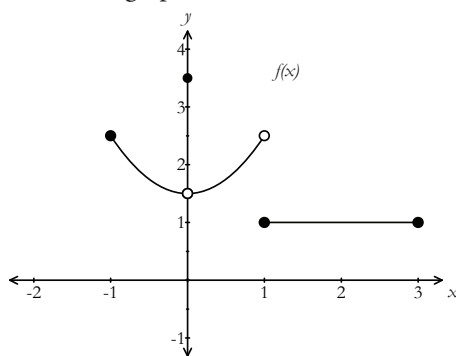
- Ask students to evaluate the limits using the following piecewise function:



- (i)  $\lim_{x \rightarrow -1^+} f(x)$                       (ii)  $\lim_{x \rightarrow 1^-} f(x)$
- (iii)  $\lim_{x \rightarrow 1^+} f(x)$                       (iv)  $\lim_{x \rightarrow 2^+} f(x)$
- (v)  $\lim_{x \rightarrow 2^-} f(x)$                       (vi)  $\lim_{x \rightarrow 2} f(x)$

(C1.4, C1.5)

- Ask students to identify which limit statements are true and which are false for the graph shown.



- (i)  $\lim_{x \rightarrow -1^+} f(x) = 2.5$                       (ii)  $\lim_{x \rightarrow 1^-} f(x) = 1$
- (iii)  $\lim_{x \rightarrow 1^+} f(x) = 1$                       (iv)  $\lim_{x \rightarrow 0^-} f(x) = 1.5$
- (v)  $\lim_{x \rightarrow 3^-} f(x) = 1$

(C1.4, C1.5)

*Journal*

- Ask students to explain how both a left hand and right hand limit can exist but the limit at that point may not exist. They should illustrate with a diagram.

(C1.4, C1.5)

**Resources/Notes****Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**1.3: The Limit of a Function**

SB: pp. 24 - 35

IG: pp. 19 - 34

PPL: SteEC\_01\_03.ppt

Video: Examples 6 and 7

## Calculus Reasoning

### Outcomes

Students will be expected to

C1 Continued ...

#### Achievement Indicators:

C1.4, C1.5 *Continued*

### Elaborations – Strategies for Learning and Teaching

Alternatively, limits can be determined using a table of values. Ask students to determine, for example,  $\lim_{x \rightarrow 3^-} f(x)$  and  $\lim_{x \rightarrow 3^+} f(x)$  for the function  $f(x) = \frac{x^2 - 9}{x - 3}$ .

They should evaluate the function for  $x$  values that get closer and closer to 3. Ask students to complete the following table to determine the  $y$  values.

$x$	$f(x) = \frac{x^2 - 9}{x - 3}$
2.9	5.9
2.99	5.99
2.999	5.999
2.9999	5.9999

$x$	$f(x) = \frac{x^2 - 9}{x - 3}$
3.1	6.1
3.01	6.01
3.001	6.001
3.0001	6.0001

From this, students should observe that:

- (i)  $\lim_{x \rightarrow 3^-} f(x) = 6$
- (ii)  $\lim_{x \rightarrow 3^+} f(x) = 6$
- (iii) Since the left and right hand limits are the same,  $\lim_{x \rightarrow 3} f(x) = 6$ .

C1.6 *Apply the properties of limits including:*

- *Sum Rule*
  - *Difference Rule*
  - *Product Rule*
  - *Constant Multiple Rule*
  - *Quotient Rule*
  - *Power Rule*
- to solve problems.*

C1.7 *Determine the value of the limit of a function as the variable approaches a real number*

- *by substitution*
- *by algebraic manipulation.*

To evaluate limits algebraically, students should first be exposed to the properties of limits. Suppose  $\lim_{x \rightarrow a} f(x)$  and  $\lim_{x \rightarrow a} g(x)$  both exist and let  $c$  be any constant.

Introduce and guide students through the following properties:

Rule Name	Rule
Constant Rule	$\lim_{x \rightarrow a} c = c$
Sum Rule	$\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
Difference Rule	$\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
Product Rule	$\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \times \lim_{x \rightarrow a} g(x)$
Constant Multiple	$\lim_{x \rightarrow a} [c \cdot f(x)] = c \lim_{x \rightarrow a} f(x)$
Quotient Rule	$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}, \quad \lim_{x \rightarrow a} g(x) \neq 0$
Power Rule	$\lim_{x \rightarrow a} [f(x)]^{\frac{m}{n}} = \left[ \lim_{x \rightarrow a} f(x) \right]^{\frac{m}{n}}$

**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Paper and Pencil*

- Provide students the following information:

$$\lim_{x \rightarrow 3} f(x) = 5$$

$$\lim_{x \rightarrow 3} g(x) = -4$$

$$\lim_{x \rightarrow 3} h(x) = 0$$

Ask them to find the following limits if they exist:

(i)  $\lim_{x \rightarrow 3} [f(x) + 6g(x)]$

(ii)  $\lim_{x \rightarrow 3} [g(x)]^2$

(iii)  $\lim_{x \rightarrow 3} \left[ \frac{g(x)}{h(x)} \right]$

(iv)  $\lim_{x \rightarrow 3} \left[ \frac{f(x) \times h(x)}{g(x)} \right]$

(C1.6)

**Resources/Notes****Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**1.4: Calculating Limits**

SB: pp. 35 - 45

IG: pp. 35 - 40

PPL: SteEC\_01\_04.ppt

## Calculus Reasoning

### Outcomes

Students will be expected to

C1 Continued...

### Achievement Indicators:

C1.6, C1.7 *Continued*

### Elaborations—Strategies for Learning and Teaching

When evaluating limits as  $x$  approaches a real value, students should solve problems involving the following two cases:

- The limit value is defined when the value of  $x$  is substituted into the function.
- The limit value is not defined when the value of  $x$  is substituted into the function (i.e., indeterminate form  $\frac{0}{0}$ ). When working with the indeterminate form, students should algebraically manipulate the expression in order to evaluate the limit.

Students should work through examples where they apply the rules of limits.

$$\begin{aligned}\lim_{x \rightarrow 2} (x^3 + 3x^2 - 1) &= \lim_{x \rightarrow 2} x^3 + 3 \lim_{x \rightarrow 2} x^2 - \lim_{x \rightarrow 2} 1 \\ &= (2)^3 + 3(2)^2 - 1 \\ &= 19\end{aligned}$$

They should observe that the use of the limit rules proves that direct substitution yields the same result when they substitute in the value of  $a$  (i.e.,  $\lim_{x \rightarrow a} f(x) = f(a)$ ).

Students should evaluate limit problems that involve a quotient of two polynomials, such as  $\lim_{x \rightarrow -1} \frac{x^2 + 3x + 7}{x + 4}$ . Ask students to first substitute in  $-1$  for  $x$ , resulting in a finite limit  $\frac{5}{3}$ . They should then verify their answer using the limit laws. Since both the numerator and denominator are polynomials, students compute the limits of the numerator and the denominator and hence the limit itself. Caution students that this is only true provided the limit of the denominator does not equal zero.

$$\begin{aligned}\lim_{x \rightarrow -1} \frac{x^2 + 3x + 7}{x + 4} &= \frac{\lim_{x \rightarrow -1} (x^2 + 3x + 7)}{\lim_{x \rightarrow -1} (x + 4)} \\ &= \frac{\lim_{x \rightarrow -1} x^2 + 3 \lim_{x \rightarrow -1} x + \lim_{x \rightarrow -1} 7}{\lim_{x \rightarrow -1} x + \lim_{x \rightarrow -1} 4} \\ &= \frac{1 - 3 + 7}{-1 + 4} \\ &= \frac{5}{3}\end{aligned}$$

Discuss with students that this example involves a combination of several of the limit properties where none of the restrictions are violated. This, however, is not always the case.



**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Performance*

- Ask students to work in groups to participate in a game, similar to dominos, called *Limit to Win-It*. Provide each group with 10 cards. One half of the card should contain a limit expression to be evaluated, while the other half will have the answer for a different limit expression. The task for students is to lay the cards out such that the answer on one card will match the correct limit expression on another. The cards will eventually form a complete loop with the first card matching with the last card. A sample is shown below:

$\lim_{x \rightarrow -2} \frac{(x+1)^2 - 1}{x+2} \quad \Bigg  \quad 6$	$\lim_{x \rightarrow 9} \frac{x-9}{\sqrt{x}-3} \quad \Bigg  \quad \frac{1}{2}$	$\lim_{x \rightarrow 1} \frac{x-1}{x^2-1} \quad \Bigg  \quad -4$
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(C1.6, C1.7)

**Resources/Notes****Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**1.4: Calculating Limits**

SB: pp. 35 - 45

IG: pp. 35 - 40

PPL: SteEC\_01\_04.ppt

Video Examples: 4 and 7

## Calculus Reasoning

### Outcomes

*Students will be expected to*

C1 Continued...

### Achievement Indicators:

C1.6, C1.7 *Continued*

C1.8 *Determine limits that result in infinity (infinite limits).*

C1.9 *Investigate the behaviour of the function at a vertical asymptote using limits.*

### Elaborations – Strategies for Learning and Teaching

Students should develop techniques to help them determine  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$  where substitution results in  $\frac{0}{0}$ . This is known as an indeterminate form. Students are aware that zero in the denominator means the function is undefined. Remind them that this is only true if the numerator is also not zero. They are also aware that zero in the numerator usually means that the fraction is zero, unless the denominator is also zero.

Use examples such as the following to introduce students to the various kinds of indeterminate forms.

$$(i) \lim_{x \rightarrow 3} \frac{x^2 - x - 6}{x - 3} \quad (ii) \lim_{x \rightarrow 4} \frac{x - 4}{\sqrt{x} - 2} \quad (iii) \lim_{x \rightarrow 10} \frac{(x - 5)^2 - 25}{x - 10}$$

As students evaluate the limit problems, they should think about the algebraic techniques that can be applied to simplify the expression so direct substitution can be used to determine the limit. Students should simplify the function where possible by

- factoring  $f(x)$  and  $g(x)$  and reducing  $\frac{f(x)}{g(x)}$  to lowest terms,
- rationalizing the numerator or denominator if  $f(x)$  or  $g(x)$  contains a square root,
- expanding  $f(x)$  or  $g(x)$ ,
- rewriting an absolute value using piecewise notation.

Students should evaluate limits whose result is infinity or negative infinity. Discuss with students that some functions increase or decrease without bound near certain values for the independent variable. When this occurs, the function is said to have an infinite limit.

In Mathematics 2200, students were exposed to rational functions where the numerator was a constant (RF11). They are aware that values of  $x$  which make the denominator zero result in vertical asymptotes. Students should now progress to rational functions where the numerator contains a polynomial. They should investigate the behaviour of the function at a vertical asymptote using limits.

A rational function has an infinite limit if the limit of the denominator is zero and the limit of the numerator is not zero. The sign of the infinite limit is determined by the sign of the quotient of the numerator and the denominator at values close to the number that the independent variable is approaching.

**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Performance*

- In groups of two, ask students to participate in the activity *Pass the Problem*. Distribute a problem to each pair that involves evaluating limits. Ask one student to write the first line of the solution and then pass it to the second student. The second student will verify the workings and check for errors. If there is an error present, ask students to discuss the error and why it occurred. The student will then write the second line of the solution and pass it to their partner. This process continues until the solution is complete.

Examples:

$$(i) \lim_{x \rightarrow -4} \frac{\frac{1}{4} + \frac{1}{x}}{4 + x} \qquad (ii) \lim_{x \rightarrow 0} \frac{(2 + x)^3 - 8}{x}$$

$$(iii) \lim_{x \rightarrow -1} \frac{x^2 - 4x}{x^2 - 3x - 4} \qquad (iv) \lim_{x \rightarrow -1} \frac{\sqrt{x^2 + 9} - 5}{x}$$

$$(v) \lim_{x \rightarrow -5} \frac{2x + 10}{|x + 5|}$$

(C1.6, C1.7)

- This activity will require about ten to fifteen limit problems that have been evaluated, cut apart, and put in separate envelopes.
  - Teachers distribute a problem and ask each group to evaluate the limit where each step of their workings is written per line.
  - Teachers should verify the solution is correct.
  - Ask students to cut apart their solution separating each step, place them in an envelope and exchange it with another group.
  - Ask students to put the limit steps back together in the correct order.
  - A recorder in the group will write the full solution for the group to submit to the teacher.

(C1.6, C1.7)

**Resources/Notes****Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**1.4: Calculating Limits**

SB: pp. 35 - 45

IG: pp. 35 - 40

PPL: SteEC\_01\_04.ppt

Video Examples: 4 and 7

**1.6: Limits Involving Infinity**

SB: pp. 56 - 69

IG: pp. 51 - 59

PPL: SteEC\_01\_06.ppt

**Suggested Resource****Resource Link:**

[www.k12pl.nl.ca/curr/10-12/math/3208/res-link/limit.html](http://www.k12pl.nl.ca/curr/10-12/math/3208/res-link/limit.html)

- classroom clip of students solving a limit puzzle

## Calculus Reasoning

### Outcomes

Students will be expected to

C1 Continued...

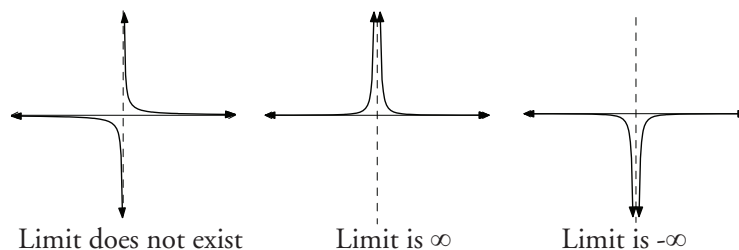
### Achievement Indicators:

C1.8, C1.9 *Continued*

### Elaborations—Strategies for Learning and Teaching

Ask students to evaluate a limit such as  $\lim_{x \rightarrow 0} \frac{1}{x^2}$ . Remind them that the limit from the left must equal the limit from the right in order for the limit to exist. Encourage students to explain their reasoning as they evaluate the limit. They should notice that, as  $x$  approaches 0 (either from the left or the right), the numerator is always positive 1, and the denominator approaches 0 through positive values. The function therefore increases without bound where  $\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$ . It is important for students to analyze the graph of  $y = \frac{1}{x^2}$  to help them make the connection that the function has a vertical asymptote at  $x = 0$ . Ask students how their answer would change if they evaluated  $\lim_{x \rightarrow 0} (\frac{1}{x})$ .

Provide students with the following graphs when discussing the behaviour on either side of a vertical asymptote.



It is important to note to students that the limits above actually do not exist, since they do not approach a specific value ( $\infty$  is not a real number). It is, however, conventional to assign either positive or negative infinity as required to indicate that the one-sided limits agree on either side of the vertical asymptote.

Students should progress to evaluating limits such as  $\lim_{x \rightarrow 2} \frac{x+4}{x-2}$ . As they evaluate the limit, their reasoning should center around the following points:

- As  $x$  approaches 2 from the left, the numerator approaches 6, and the denominator approaches 0 through negative values:  

$$\lim_{x \rightarrow 2^-} \frac{x+4}{x-2} = -\infty.$$
- As  $x$  approaches 2 from the right, the numerator approaches 6, and the denominator approaches 0 through positive values:  $\lim_{x \rightarrow 2^+} \frac{x+4}{x-2} = \infty$ .
- The  $\lim_{x \rightarrow 2} \frac{x+4}{x-2}$  does not exist since the one-sided limits do not exist.
- The function has a vertical asymptote at  $x = 2$ .

Encourage students to observe the graph of  $y = \frac{x+4}{x-2}$  to verify their results.

**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Journal*

- Ask students to respond to the following:

- (i) If  $f(x)$  has a vertical asymptote at  $x = 3$ , would the  $\lim_{x \rightarrow 3} f(x)$  have to exist? Illustrate your response with a diagram.

(C1.8, C1.9)

- (ii) Find the vertical asymptotes of the function  $y = \frac{1}{x^2 - 4}$  and describe the behaviour of the function to the left and to the right of each vertical asymptote.

(C1.8, C1.9)

- (iii) Investigate the behaviour of  $f(x) = \frac{3x^2 + 5x - 2}{9x^2 - 1}$  at any vertical asymptotes.

(C1.8, C1.9)

**Resources/Notes****Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**1.6: Limits Involving Infinity**

SB: pp. 56 - 69

IG: pp. 51 - 59

PPL: SteEC\_01\_06.ppt

## Calculus Reasoning

### Outcomes

*Students will be expected to*

C1 Continued...

#### Achievement Indicators:

C1.8, C1.9 *Continued*

C1.10 *Evaluate limits of functions as  $x$  approaches infinity (limits at infinity).*

C1.11 *Investigate the end behaviour of the function using limits to identify possible horizontal asymptotes.*

### Elaborations – Strategies for Learning and Teaching

As students work through an example similar to  $\lim_{x \rightarrow 0^+} (\frac{1}{x} - \frac{1}{x^2})$ , remind them to rewrite  $\frac{1}{x} - \frac{1}{x^2}$  as an equivalent fractional expression  $\frac{x-1}{x^2}$ . They should notice as  $x$  approaches 0 from the right, the numerator approaches -1 and the denominator approaches 0 through positive values. The function decreases without bound resulting in  $\lim_{x \rightarrow 0^+} \frac{x-1}{x^2} = -\infty$  where the vertical asymptote of the graph is  $x = 0$ .

Discuss the following points with students:

- $\lim_{x \rightarrow 0^+} (\frac{1}{x} - \frac{1}{x^2}) \neq \lim_{x \rightarrow 0^+} (\frac{1}{x}) - \lim_{x \rightarrow 0^+} (\frac{1}{x^2})$  since individually they do not exist.
- $\infty - \infty$  does not equal zero (i.e., because  $\pm\infty$  are not real numbers the subtraction has no meaning).

Students will continue to work with infinite limits later in this course when they graph rational functions. They will determine vertical asymptotes by evaluating the limit of the function as  $x$  approaches a value that makes the denominator of the function equal to zero. If either  $\lim_{x \rightarrow a^-} f(x) = \pm\infty$  or  $\lim_{x \rightarrow a^+} f(x) = \pm\infty$ , they should recognize the line  $x = a$  is a vertical asymptote of the graph of the function  $y = f(x)$ .

Limits at infinity are used to describe the behaviour of functions as the independent variable increases or decreases without bound. As students investigate the end behaviour of the graph of a function (i.e., what happens when the value of  $x$  approaches  $\pm\infty$ ), they should also make the connection that limits at infinity provide information about the equation of any existing horizontal asymptotes.

Students previously established that  $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$ . This concept should now be expanded to include  $\lim_{x \rightarrow \infty} \frac{1}{x^n} = 0$ .

The most common method used to evaluate this type of limit is to divide all terms by the highest power of  $x$ , and then apply  $\lim_{x \rightarrow \infty} \frac{1}{x^n} = 0$ .

When evaluating  $\lim_{x \rightarrow \infty} \frac{x^3 - 6x}{2x^4 + 1}$ , for example, students should divide each term by  $x^4$ . Ensure they make the connection that the value of the limit represents the horizontal asymptote  $y = 0$ .

Teachers should emphasize that horizontal asymptotes involve the end behaviour of the function. The graph of a function may in fact cross the horizontal asymptote at other values of the independent variable.

**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Paper and Pencil*

- Ask students to evaluate  $\lim_{x \rightarrow \infty} f(x)$  for each of the following functions. They should identify the equation of the horizontal asymptote of the graph if it exists.

(i)  $f(x) = \frac{3x^2 - 2x + 4}{2x^2 - 1}$

(ii)  $f(x) = \frac{4x^3 - 2x + 1}{x + 2}$

(iii)  $f(x) = \frac{x^2 - x + 5}{4x^3 + 2x^2 - x}$

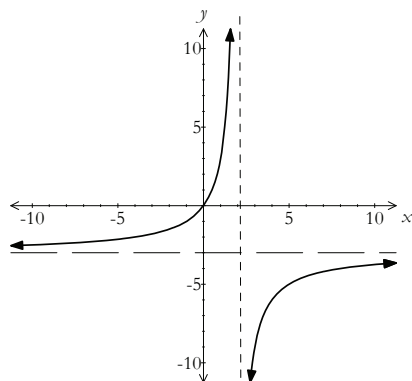
(C1.10, C1.11)

*Performance*

- In groups of two, ask students to produce the graph of a rational function using limit characteristics. Give one student a list of limit characteristics along with the desired graph. Ask him or her to read the list of characteristics to their partner. The goal of the other student is to create the graph being described. Both students will then check to see how well their graphs match. A sample is shown below:

Given:  $\lim_{x \rightarrow 2^-} f(x) = \infty$ ,  $\lim_{x \rightarrow \infty} f(x) = -3$  and  $\lim_{x \rightarrow 2^+} f(x) = -\infty$

Possible Graph:



(C1.8, C1.9, C1.10, C1.11)

**Resources/Notes****Authorized Resource***Single Variable Essential Calculus  
Second Edition***1.6: Limits Involving Infinity**

SB: pp. 56 - 69

IG: pp. 51 - 59

PPL: SteEC\_01\_06.ppt

Video Examples: 5

## Calculus Reasoning

### Outcomes

Students will be expected to

C1 Continued...

### Achievement Indicators:

C1.10, C1.11 *Continued*

### Elaborations – Strategies for Learning and Teaching

Horizontal asymptotes can also be found using the degree of the numerator and denominator. Using graphing technology, ask students to compare the horizontal asymptotes for each rational function:

$$f(x) = \frac{x^2 - 4x}{x^2 - 3x - 4}, \quad f(x) = \frac{x^3 - 6x}{x^4 + 1}, \quad f(x) = \frac{x^2 - 2x - 3}{x + 3}$$

Students should observe the following patterns:

- If the degree of the numerator is less than the degree of the denominator, the horizontal asymptote is  $y = 0$ .
- If the degree of the numerator is equal to the degree of the denominator, the horizontal asymptote is the quotient of the leading coefficients.
- If the degree of the numerator is greater than the degree of the denominator, there is no horizontal asymptote.

When working with limits at infinity, students should be exposed to functions similar to  $f(x) = \frac{\sqrt{x^2 + 1}}{2x - 3}$ . When determining the vertical asymptote, remind them to identify where the function is undefined. Ask students to now focus on the horizontal asymptotes and determine how many exist. Guide them through the process by calculating the values of the following limits:

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 1}}{2x - 3} \qquad \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2 + 1}}{2x - 3}$$

In Mathematics 2200, students were exposed to  $\sqrt{x^2} = x$  where  $x \geq 0$  and  $\sqrt{x^2} = -x$  where  $x < 0$  (AN2).

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 1}}{2x - 3} &= \lim_{x \rightarrow \infty} \frac{\sqrt{x^2} \sqrt{1 + \frac{1}{x^2}}}{2x - 3} & \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2 + 1}}{2x - 3} &= \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2} \sqrt{1 + \frac{1}{x^2}}}{2x - 3} \\ &= \lim_{x \rightarrow \infty} \frac{\sqrt{x^2} \sqrt{1 + \frac{1}{x^2}}}{2x - 3} & &= \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2} \sqrt{1 + \frac{1}{x^2}}}{2x - 3} \\ &= \lim_{x \rightarrow \infty} \frac{x \sqrt{1 + \frac{1}{x^2}}}{x(2 - \frac{3}{x})} & &= \lim_{x \rightarrow -\infty} \frac{-x \sqrt{1 + \frac{1}{x^2}}}{x(2 - \frac{3}{x})} \\ &= \lim_{x \rightarrow \infty} \frac{\sqrt{1}}{(2)} & &= \lim_{x \rightarrow -\infty} \frac{-\sqrt{1}}{(2)} \\ &= \frac{1}{2} & &= -\frac{1}{2} \end{aligned}$$

Students should conclude there are two horizontal asymptotes:  $y = \frac{1}{2}$  and  $y = -\frac{1}{2}$ .



---

**General Outcome: Develop introductory calculus reasoning.**

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**Suggested Assessment Strategies**

*Paper and Pencil*

- Ask students to determine the vertical and horizontal asymptotes of

(i)  $f(x) = \frac{8x-1}{\sqrt{4x^2+x+6}}$

(ii)  $f(x) = \frac{\sqrt{2x^2+1}}{3x-5}$

(C1.10, C1.11)

**Resources/Notes****Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**1.6: Limits Involving Infinity**

SB: pp. 56 - 69

IG: pp. 51 - 59

PPL: SteEC\_01\_06.ppt

Video: Example 5

## Calculus Reasoning

## Outcomes

Students will be expected to

C1 Continued...

## Achievement Indicator:

C1.12 Investigate the end behaviour of the function using limits to identify possible oblique asymptotes.

## Elaborations – Strategies for Learning and Teaching

Oblique asymptotes, or slant asymptotes, occur in rational functions where the degree of the numerator is one more than the degree of the denominator. An oblique asymptote exists for a rational function,  $f(x)$ , as described, if  $\lim_{x \rightarrow \pm\infty} [f(x) - (mx + b)] = 0$ . The line  $y = mx + b$  is called the oblique asymptote. Teachers should discuss that this limit is expressing that the vertical distance between the graph of  $f(x)$  and the line  $y = mx + b$  is approaching 0 as  $x$  approaches  $\pm\infty$ .

It is worth noting that this condition does not always result in an oblique asymptote. If, for example,

$$f(x) = \frac{x^2 + x - 2}{x + 2}, \text{ then}$$

$$\lim_{x \rightarrow \pm\infty} \frac{x^2 + x - 2}{x + 2} = \lim_{x \rightarrow \pm\infty} \frac{(x + 2)(x - 1)}{(x + 2)} = \lim_{x \rightarrow \pm\infty} (x - 1) = \pm\infty.$$

In this example, there is no oblique asymptote as the function simplifies to a linear graph with a point of discontinuity at  $(-2, -3)$ .

Students need to be able to change a rational function into the form  $f(x) - (mx + b)$ . They must divide the numerator by the denominator using polynomial long division. The concepts of polynomial long division and synthetic division were introduced to students in Mathematics 3200 (RF10). It may be beneficial to first work with a numerical example and then progress to an algebraic expression and have students write the division statement.

Numerical Example:

$$\begin{array}{r} 18 \\ 23 \overline{)435} \\ \underline{23} \\ 205 \\ \underline{184} \\ 21 \text{ Remainder} \end{array}$$

Division Statement:

$$\frac{435}{23} = 18 + \frac{21}{23}$$

Algebraic Example:

$$\begin{array}{r} x + 1 \\ x - 3 \overline{)x^2 - 2x - 8} \\ \underline{-(x^2 - 3x)} \\ x - 8 \\ \underline{-(x - 3)} \\ -5 \text{ Remainder} \end{array}$$

Division Statement:

$$\frac{x^2 - 2x - 8}{x - 3} = x + 1 + \frac{-5}{x - 3}$$

$$\frac{x^2 - 2x - 8}{x - 3} - (x + 1) = \frac{-5}{x - 3}$$

Since  $\lim_{x \rightarrow \infty} \left[ \frac{x^2 - 2x - 8}{x - 3} - (x + 1) \right] = \lim_{x \rightarrow \infty} \frac{-5}{x - 3}$ , students only need to work out the right hand side.

Therefore, because  $\lim_{x \rightarrow \infty} \frac{-5}{x - 3} = 0$ , the line  $y = x + 1$  is an oblique asymptote for the function. In cases where the rational expression has a linear denominator, synthetic division may be used. Students should be encouraged to use graphing technology to explore all aspects of oblique asymptotes.

**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Paper and Pencil*

- Ask students to determine the equation of the oblique asymptote for the following:

(i)  $f(x) = \frac{x^2 - x + 2}{x - 3}$

(ii)  $f(x) = \frac{2x^3 + 4x^2 - 9}{3 - x^2}$

(iii)  $f(x) = \frac{x^3 + 4x^2 + 16}{x^2 + 4}$

(iv)  $f(x) = \frac{3x^4 - x^3 + 6x^2 + 5x - 2}{x^3 + x + 5}$

(C1.12)

- Alicia rewrote  $f(x) = \frac{-3x^2 + 2}{x - 1}$  as  $f(x) = -3x - 3 + \frac{-1}{x - 1}$ .

Using this information, ask students how they can determine the equation of the oblique asymptote? What is the equation of the oblique asymptote?

(C1.12)

*Performance*

- Ask students to create a rational function where the degree of the numerator is one more than the degree of the denominator and write it on a sticky note along with their name. They should then write the oblique asymptote on the other side. Placing the sticky notes on the wall, students should randomly pick another note and write the oblique asymptote. They check the back of the note to verify their answer. If it is incorrect, they should discuss their answer with the student who created the function.

(C1.12)

**Resources/Notes****Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**3.4: Curve Sketching**

SB: pp. 166 - 173

IG: pp. 25 - 34

PPL: SteEC\_03\_04.ppt

**Note:**

Questions #39 - 42 in the student book on p. 173 asks students to determine the slant asymptote of a rational function and use it to help sketch the graph. Graphing will be developed in the next unit, so for this question students are only expected to determine the equation of the slant asymptote.

## Calculus Reasoning

### Outcomes

Students will be expected to

C2 Solve problems involving continuity.

[C, CN, R, T, V]

#### Achievement Indicators:

C2.1 Distinguish between the concepts of continuity and discontinuity of a function informally.

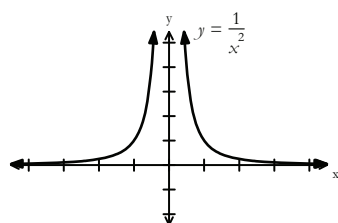
C2.2 Identify examples of discontinuous functions and the types of discontinuities they illustrate, such as removable, infinite, jump, and oscillating discontinuities.

### Elaborations – Strategies for Learning and Teaching

The concepts of continuity and discontinuity are new for students. A good way to introduce this idea is to have students print each letter of the alphabet using upper case. After all 26 letters have been printed, ask them how many they were able to complete without lifting their pencil from the paper. These letters would be considered “continuous” and the others “discontinuous”. The same idea can apply when graphing functions; if the graph can be drawn without lifting the pencil from the paper then it is continuous, otherwise discontinuous.

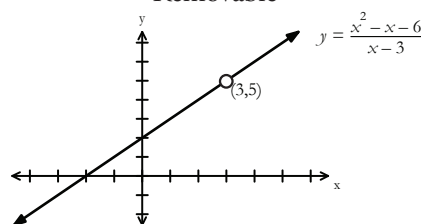
There are several types of discontinuities, including removable, infinite, jump, and oscillating discontinuities. The following graphs could be used to discuss the various types of discontinuities. Ask students to first predict why they think the graph is discontinuous. Teachers can then share the following responses with them.

Infinite



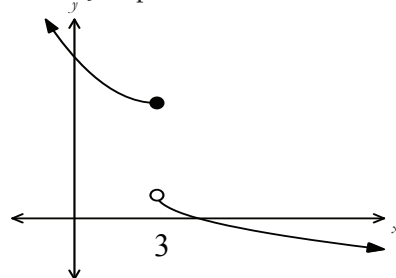
the one sided limits of the function at  $x = 0$  are infinite; graphically this situation corresponds to a vertical asymptote.

Removable



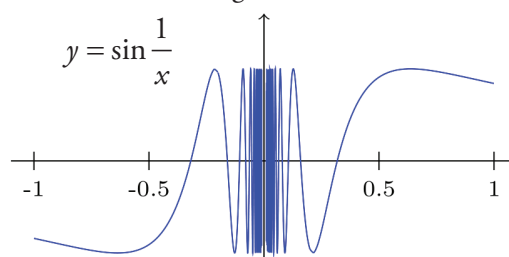
$f(x)$  is not defined at the point  $x = 3$ ; a value can be assigned to  $f(3)$  to make the extended function continuous at  $x = 3$ .

Jump



The left and right hand limits at  $x = 3$  are not equal. Therefore, the limit does not exist.

Oscillating



The function  $f(x)$  is not defined at  $x = 0$  so it is not continuous at  $x = 0$ . The function also oscillates between  $-1$  and  $1$  as  $x$  approaches  $0$ . Therefore, the limit does not exist.

---

**General Outcome: Develop introductory calculus reasoning.**

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**Suggested Assessment Strategies***Observation*

- Students could be provided with four laminated cards representing removable, infinite, jump or oscillating discontinuity, respectively. Display various graphs illustrating the different types of discontinuities and ask students to hold up the card reflecting their choice.

(C2.1, C2.2)

**Resources/Notes****Authorized Resource***Single Variable Essential Calculus  
Second Edition***1.5: Continuity**

SB: pp. 46 - 56

IG: pp. 41 - 50

PPL: SteEC\_01\_05.ppt

Video Examples: 2

## Calculus Reasoning

### Outcomes

Students will be expected to

C2 Continued...

#### Achievement Indicators:

C2.3 Determine whether a function is continuous at a point from its graph.

C2.4 Determine whether a function is continuous at a point using the definition of continuity.

### Elaborations—Strategies for Learning and Teaching

Students should be introduced to the formal definition of continuity. Inform them of the three conditions necessary for a function to be continuous at a particular value of  $x$ .

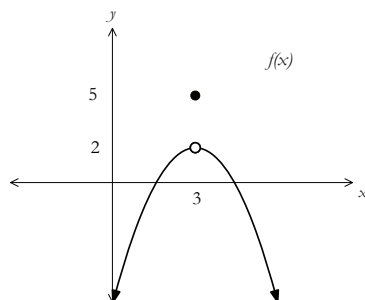
A function  $f$  is continuous at  $x = a$  when:

- (i)  $f(a)$  is defined
- (ii)  $\lim_{x \rightarrow a} f(x)$  exists, which implies  $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x)$
- (iii)  $\lim_{x \rightarrow a} f(x) = f(a)$

If any one of the above conditions is not met, the function is discontinuous at  $x = a$ .

When students are asked to determine whether a function is continuous at a point, it would be a good idea for them to first analyze a graph.

Ask students to determine if  $f(x)$  is continuous at  $x = 3$ .



They should make the following observations:

- (i)  $f(3) = 5$
- (ii)  $\lim_{x \rightarrow 3} f(x) = 2$
- (iii)  $\lim_{x \rightarrow 3} f(x) = 2 \neq f(3)$

The limit of the function exists but does not equal the value of the function at that point. Therefore,  $f(x)$  is discontinuous at  $x = 3$ .

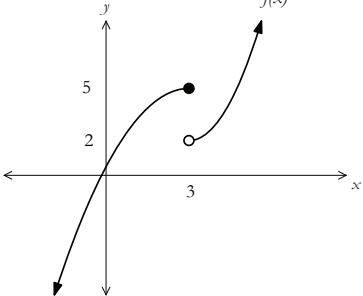
## General Outcome: Develop introductory calculus reasoning.

### Suggested Assessment Strategies

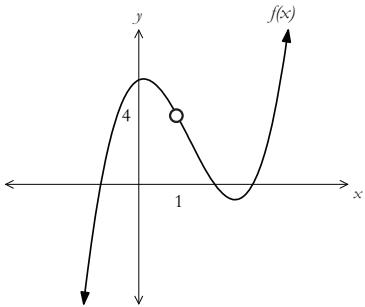
#### Interview

- Ask students to analyze the following graphs and determine whether the functions are continuous at the indicated point. They should explain their reasoning.

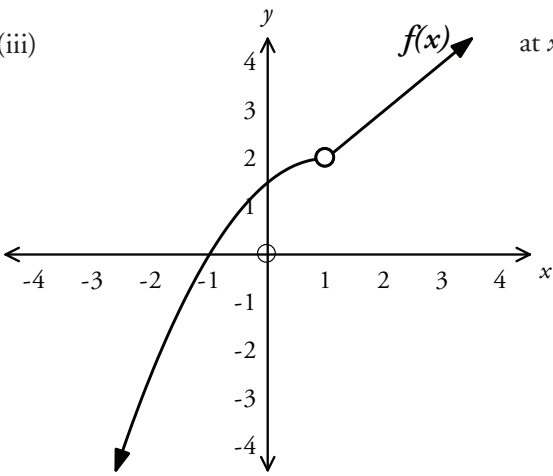
(i)  at  $x = 3$ ?



(ii)  at  $x = 1$ ?



(iii)  at  $x = 1$ ?



(C2.3, C2.4)

### Resources/Notes

#### Authorized Resource

*Single Variable Essential Calculus  
Second Edition*

#### 1.5: Continuity

SB: pp. 46 - 56

IG: pp. 41 - 50

PPL: SteEC\_01\_05.ppt

Video Examples: 2

## Calculus Reasoning

## Outcomes

Students will be expected to

C2 Continued...

## Achievement Indicators:

C2.3, C2.4 Continued

## Elaborations – Strategies for Learning and Teaching

When determining continuity, students should move from a visual representation to an algebraic approach. Ask students to determine if the following piecewise function is continuous at  $x = 3$ :

$$f(x) = \begin{cases} 3x - 2, & x < 3 \\ x^2 + x - 8, & x \geq 3 \end{cases}$$

Students must choose the correct function when evaluating  $f(3)$ ,  $\lim_{x \rightarrow 3^-} f(x)$  and  $\lim_{x \rightarrow 3^+} f(x)$ . Ensure they follow the conditions for continuity:

- (i)  $f(3) = 4$
- (ii)  $\lim_{x \rightarrow 3^-} f(x) = 7$  and  $\lim_{x \rightarrow 3^+} f(x) = 4$
- (iii) The  $\lim_{x \rightarrow 3} f(x)$  does not exist and the function is discontinuous at  $x = 3$ .

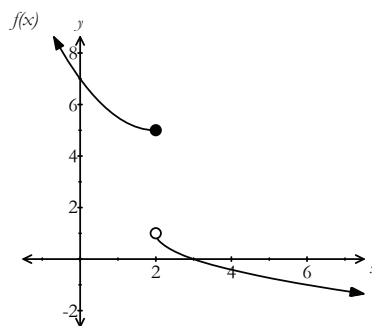
Students should observe that, since  $\lim_{x \rightarrow 3^+} f(x) = f(3) = 4$ , the function is continuous from the right at 3.

A function is said to be continuous from the right at  $a$  if  $\lim_{x \rightarrow a^+} f(x) = f(a)$ .

Similarly, the function is continuous from the left at  $a$  if  $\lim_{x \rightarrow a^-} f(x) = f(a)$ .

C2.5 Determine whether a function is continuous on a closed interval.

If a function is continuous at every value in a closed interval, then the function is continuous in that interval. Provide students the following graph and ask them on what interval is the function continuous.



Students should recognize that  $f(x)$  is discontinuous at  $x = 2$ , but continuous on intervals that do not include  $x = 2$ .



**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Paper and Pencil*

- Ask students to answer the following:

(i) Find the value of  $a$  so that the function  $f(x) = \begin{cases} x^2 - 1, & x < 3 \\ 2ax, & x \geq 3 \end{cases}$  is continuous.

(ii) Find the value of  $a$  so that the function  $f(x) = \begin{cases} 2x + 3, & x \leq 2 \\ ax + 1, & x > 2 \end{cases}$  is continuous.

- (iii) Determine where the function is continuous for the following functions:

(a)  $f(x) = \frac{1}{x-5}$

(b)  $f(x) = x^3 + 2x^2 - 4x + 1$

(C2.3, C2.4)

**Resources/Notes****Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**1.5: Continuity**

SB: pp. 46 - 56

IG: pp. 41 - 50

PPL: SteEC\_01\_05.ppt

## Calculus Reasoning

### Outcomes

*Students will be expected to*

C2 Continued...

### Achievement Indicator:

C2.6 Rewrite removable discontinuities by extending or modifying a function.

### Elaborations – Strategies for Learning and Teaching

Ask students to identify where the rational function, such as  $f(x) = \frac{x^2 - 3x - 4}{x - 4}$ , is discontinuous. As they factor the expression, the graph of  $f(x)$  is a straight line but with a removable discontinuity (hole) at  $x = 4$ . The function is therefore discontinuous at  $x = 4$ , but continuous elsewhere.

Inform students that a removable discontinuity occurs when there is a limit, which is finite, but the value of the function at that point either does not exist or is different to the value of the limit.

Ask students to modify the function so that it is continuous by defining  $f(4) = 5$ . The modified function can be written as:

$$f(x) = \begin{cases} \frac{x^2 - 3x - 4}{x - 4}, & x \neq 4 \\ 5, & x = 4 \end{cases}$$

---

**General Outcome: Develop introductory calculus reasoning.**

---

**Suggested Assessment Strategies**

*Paper and Pencil*

- For each of the following, ask students to provide a formula for the extended function that is continuous at the given value:

(i)  $y = \frac{x^2 - x - 2}{x - 2}$ , at  $x = 2$

(ii)  $y = \frac{x - 4}{\sqrt{x} - 2}$ , at  $x = 4$

(C2.6)

**Resources/Notes**

**Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**1.5: Continuity**

SB: pp. 46 - 56

IG: pp. 41 - 50

PPL: SteEC\_01\_05.ppt



# Rational Functions

Suggested Time: 6 Hours

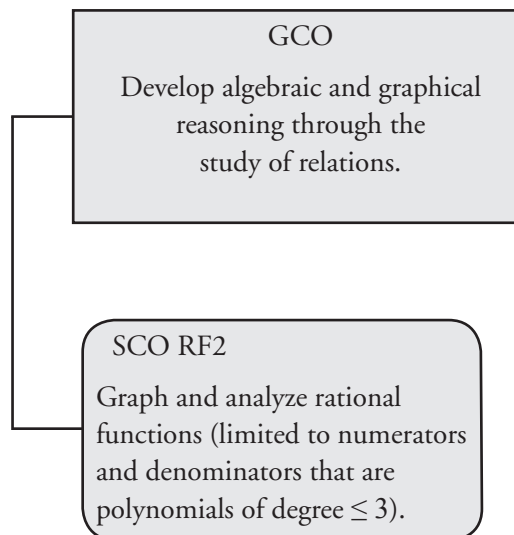
# Unit Overview

## Focus and Context

In this unit, students work with rational functions of the form  $y = \frac{f(x)}{g(x)}$  where both  $f(x)$  and  $g(x)$  are polynomials. Although polynomials are defined for all real values of  $x$ , rational functions are not defined for those values of  $x$  for which the denominator,  $g(x)$ , is zero. The  $x$ -intercepts are the zeroes of the numerator,  $f(x)$ , since the function is zero only when its numerator is zero. Students graph rational functions using intercepts, asymptotes and points of discontinuity. They determine the vertical and horizontal asymptotes of the function and verify these values using limits.

Later in this course, students will use information from the first and second derivative, combined with the characteristics of  $x$ -intercepts,  $y$ -intercept, asymptotes and points of discontinuity to refine the graph of a rational function.

## Outcomes Framework



## SCO Continuum

Mathematics 2200	Mathematics 3200	Calculus 3208
Relations and Functions	Relations and Functions	Relations and Functions
<p>RF7 Solve problems that involve linear and quadratic inequalities in two variables. [C, PS, T, V]</p> <p>RF8 Solve problems that involve quadratic inequalities in one variable. [CN, PS, V]</p> <p>RF11 Graph and analyze reciprocal functions (limited to the reciprocal of linear and quadratic functions). [CN, R, T, V]</p>	<p>RF10 Demonstrate an understanding of factoring polynomials of degree greater than 2 (limited to polynomials of degree <math>\leq 5</math> with integral coefficients). [C, CN, ME]</p> <p>RF11 Graph and analyze polynomial functions (limited to polynomials of degree <math>\leq 5</math>). [C, CN, T, V]</p>	<p>RF2 Graph and analyze rational functions (limited to numerators and denominators that are polynomials of degree <math>\leq 3</math>). [CN, R, T, V]</p>

## Mathematical Processes

[C] Communication	[PS] Problem Solving
[CN] Connections	[R] Reasoning
[ME] Mental Mathematics and Estimation	[T] Technology
	[V] Visualization

## Relations and Functions

### Outcomes

*Students will be expected to*

**RF2 Graph and analyze rational functions (limited to numerators and denominators that are polynomials of degree  $\leq 3$ ).**

[CN, R, T, V]

### Achievement Indicators:

RF2.1 *Explain the behaviour of the graph of a rational function for values of the variable near a non-permissible value.*

RF2.2 *Determine if the graph of a rational function will have an asymptote or a hole for a non-permissible value.*

RF2.3 *Sketch the graph of a rational function.*

### Elaborations – Strategies for Learning and Teaching

Students graph and analyze rational functions limited to numerators and denominators that are monomials, binomials or trinomials. They extend their knowledge of non-permissible values to distinguish between points of discontinuity and vertical asymptotes, and how these are displayed graphically.

In Mathematics 2200, students were exposed to rational functions of the form  $\frac{1}{f(x)}$ , where  $f(x)$  was a linear and a quadratic function (RF11). Review the concept of non-permissible values and how these values resulted in vertical asymptotes for the graph of  $\frac{1}{f(x)}$ .

Students should now work with rational functions of the form  $\frac{f(x)}{g(x)}$ . Characteristics such as intercepts, points of discontinuity, asymptotes and end behaviour should be discussed. It is important for students to continue to use limits to verify the equations of the vertical, horizontal, and oblique asymptotes.

Ask students to explain why only the numerator is considered when determining the  $x$ -intercepts of a rational function. Ensure they verify that this value of  $x$  does not result in a denominator of zero.

Students should work with rational functions, such as,  $f(x) = \frac{x^2 - 4x}{x^2 - 3x - 4}$  where a common factor exists in the numerator and denominator.

Using its graph as a visual representation, students should observe that there is a point on the graph where the function is undefined but there is no vertical asymptote at that point (i.e.,  $x = 4$ ). They were exposed to points of discontinuity in the previous unit when they worked with continuity and piecewise functions. They are now expected to determine these points algebraically.

Using graphing technology, ask students to graph and analyze a variety of rational functions to help them make the following observations. Remind them to first simplify the rational expression by factoring the numerator and denominator.

- When a graph has a hole in it (i.e., point of discontinuity), the numerator and denominator of the original function contain a common factor.
- A factor of only the denominator corresponds to a vertical asymptote.
- A factor of only the numerator corresponds to an  $x$ -intercept.

With reference to the function  $f(x) = \frac{x^2 - 4x}{x^2 - 3x - 4}$ , students should notice the function has a vertical asymptote at  $x = -1$ , a point of discontinuity at  $x = 4$ , and  $x$ -intercept at 0.



## General Outcome: Develop algebraic and graphical reasoning through the study of relations.

### Suggested Assessment Strategies

#### Observation

- Provide groups of seven students with a rational function. Each person in the group will be given a card with the following headings:  $x$ -intercepts,  $y$ -intercept, vertical asymptote(s), horizontal asymptote, oblique asymptote, point of discontinuity, sketch of the graph. Ask each person to discuss the characteristic they have, how to find it, and then write their answer on the card. When the problem is finished, the group should present their answer to the class.

(RF2.1, RF2.2, RF2.3)

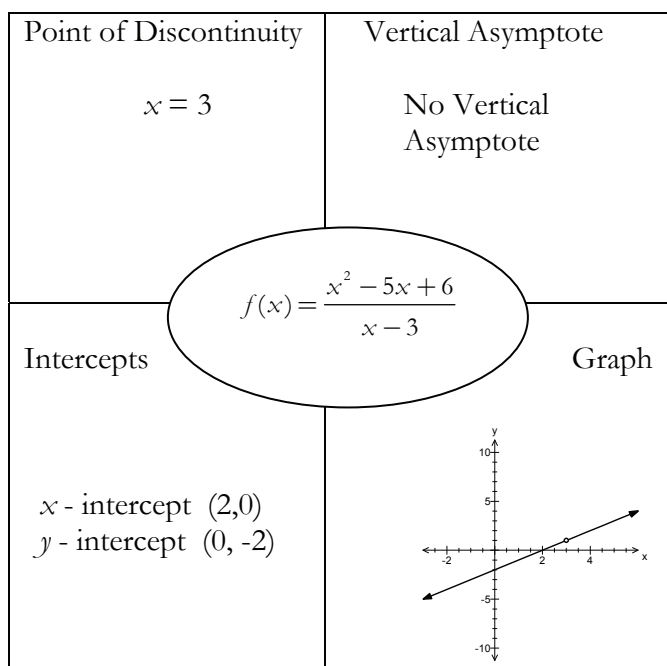
#### Journal

- Provide students the following functions  $f(x) = x^2 - 4$  and  $g(x) = x + 2$ . Ask students to explain why the new function  $h(x) = \frac{g(x)}{f(x)}$  has non-permissible values resulting in a vertical asymptote or a point of discontinuity. They should illustrate this by graphing  $h(x)$ .

(RF2.2)

#### Performance

- Students can work in pairs to complete the following rational puzzle investigating the characteristics and graphs of various rational functions. They should work with 20 puzzle pieces (4 complete puzzles consisting of a function and four related characteristics) to correctly match the characteristics with each function. A sample is shown below:



(RF2.1, RF2.2, RF2.3)

### Resources/Notes

#### Supplementary Resource

*Pre-Calculus 12*

#### 9.2: Analyzing Rational Functions

Student Book (SB): pp. 446 - 456

Teacher Resource (TR): pp. 241 - 252

#### Notes:

- Students are not expected to write the equation given the graph.
- Students will not graph rational functions using transformations. They will graph rational functions using intercepts, points of discontinuity, asymptotes, limits, sign diagram and test points. Later in this course, students will use limits and derivatives to graph rational and polynomial functions.

## Relations and Functions

### Outcomes

Students will be expected to

RF2 Continued...

#### Achievement Indicators:

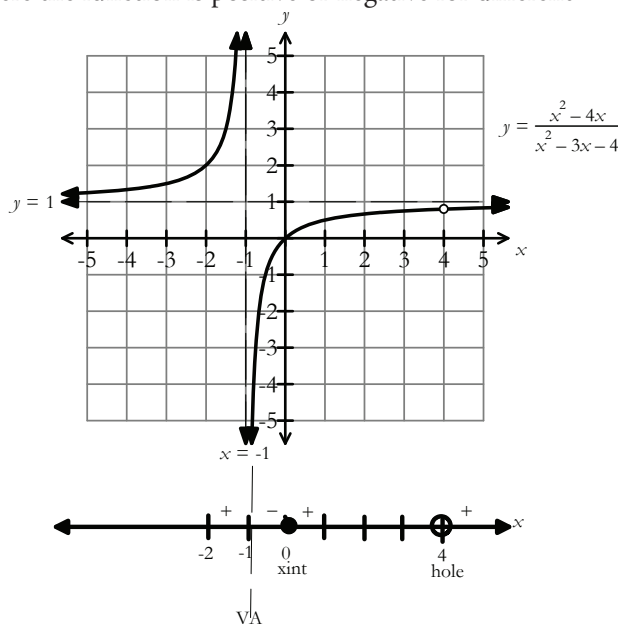
RF2.1, RF2.2, RF2.3

Continued

### Elaborations—Strategies for Learning and Teaching

Students were introduced to the sign diagram and test points in Mathematics 2200 when they solved quadratic inequalities in one variable (RF8). Sign diagrams were also used in Mathematics 3200 when graphing polynomials (RF11). They should continue to use this strategy when graphing rational functions.

Ask students to label on their sign diagram any points of discontinuity,  $x$ -intercepts or vertical asymptotes. They should then use test points to determine where the function is positive or negative for different intervals of  $x$ .



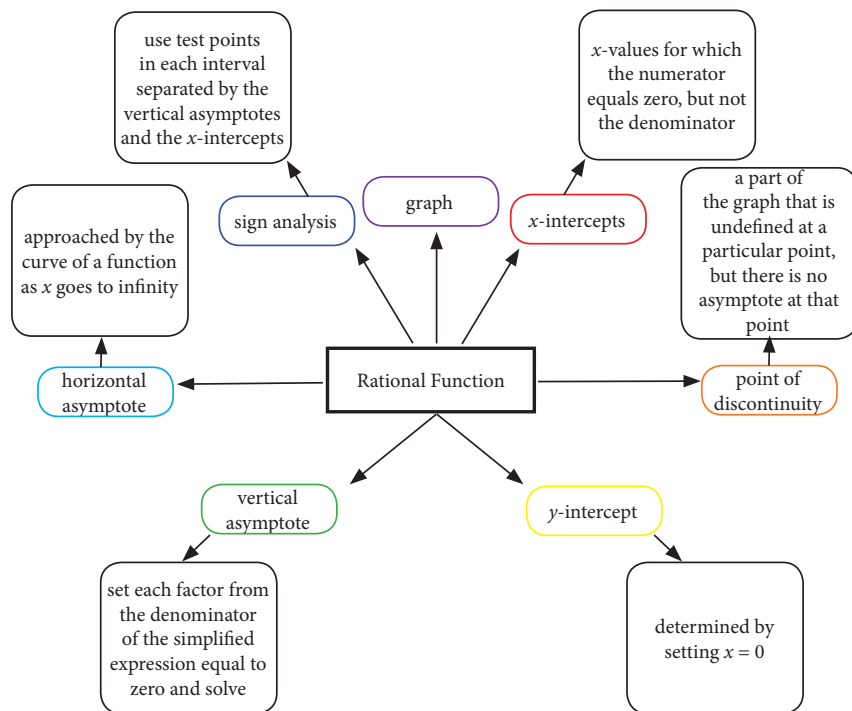
- Since the degree of the numerator and denominator are equal, students should notice that a horizontal asymptote exists. Encourage them to relate the asymptotes of the graph to the limit. Students should notice that as  $x$  approaches positive or negative infinity, the graph approaches the line given by  $y = 1$ . That is,  $\lim_{x \rightarrow \infty} f(x) = 1$  and  $\lim_{x \rightarrow -\infty} f(x) = 1$ .
- To find the  $y$ -coordinate of the point of discontinuity at  $x = 4$ , students should evaluate the limit of the simplified function:  $\lim_{x \rightarrow 4} \frac{x}{x+1}$ .
- Students should investigate the behaviour of the function at the vertical asymptote  $x = -1$ . The  $\lim_{x \rightarrow -1^+} f(x) = -\infty$  and  $\lim_{x \rightarrow -1^-} f(x) = \infty$ .
- Ask students to choose test points such as  $x = -2$ ,  $x = 2$ ,  $x = 5$  to create a more accurate graph. Later in this course, they will be exposed to the first and second derivative tests which will refine the graph even further.

**General Outcome: Develop algebraic and graphical reasoning through the study of relations.**

### Suggested Assessment Strategies

#### Performance

- Ask students to create a mind map outlining the steps to sketch the graph of a rational function. The following key points should be included:



(RF2.1, RF2.2, RF2.3)

#### Paper and Pencil

- Ask students to graph the functions using intercepts, points of discontinuity, asymptotes, limits, sign diagrams and test points.

(i)  $y = \frac{x^2+x-2}{x^2+x-20}$

(ii)  $y = \frac{x^2-5x+6}{x^2-5x+4}$

(iii)  $y = \frac{x-1}{x^2-x-6}$

(iv)  $y = \frac{9}{x^2-9}$

(v)  $y = \frac{x^3}{x^2+1}$

(RF2.1, RF2.2, RF2.3)

### Resources/Notes

#### Supplementary Resource

*Pre-Calculus 12*

#### 9.2: Analyzing Rational Functions

SB: pp. 446 - 456

TR: pp. 241 - 252

#### Suggested Resource

Resource Link:

[www.k12pl.nl.ca/curr/10-12/math/3208/res-link/rat-funs.html](http://www.k12pl.nl.ca/curr/10-12/math/3208/res-link/rat-funs.html)

- mind map template

## Relations and Functions

### Outcomes

Students will be expected to

RF2 Continued...

### Achievement Indicators:

RF2.1, RF2.2, RF2.3

*Continued*

### Elaborations – Strategies for Learning and Teaching

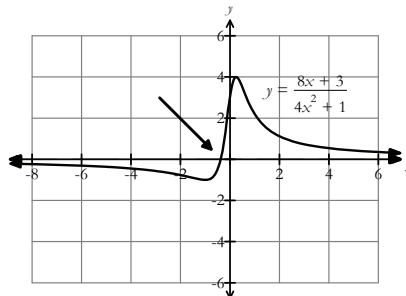
Now that students are familiar with the fact that certain rational functions have a horizontal asymptote, this would be a good time to discuss the fact that at times the rational function can indeed cross the horizontal asymptote.

Provided students a function such as  $y = \frac{8x+3}{4x^2+1}$ . Ask them the following questions:

- What is the horizontal asymptote?
- What is the domain?
- Is the function continuous?

Ask students to construct a table of values using the following  $x$ -values and then sketch its graph.

$x$	-100	-10	-1	0	1	10	100
$y$	-0.02447	-0.19202	-1	3	2.2	0.20698	0.020074



The focus here is that this function crosses its horizontal asymptote. According to the definition, a horizontal asymptote exists when  $f(x)$  approaches a line  $y = L$  as  $x$  approaches infinity. Therefore, a function may cross its horizontal asymptote many times as  $x$  approaches infinity. Students should observe that  $\lim_{x \rightarrow \infty} \frac{8x+3}{4x^2+1}$  is getting close to zero through positive numbers while  $\lim_{x \rightarrow -\infty} \frac{8x+3}{4x^2+1}$  is getting close to zero through negative numbers. This concept will be explored again when students are introduced to curve sketching using derivatives.

It is not the expectation that students be expected to find the points of intersection between the function and the horizontal asymptote but rather to be able to recognize the behaviour of the function as it approaches the horizontal asymptote.

**General Outcome: Develop algebraic and graphical reasoning through the study of relations.**

### Suggested Assessment Strategies

#### *Paper and Pencil*

- Ask students to answer the following:

(i) Sketch the graph of  $y = \frac{x(x-3)^2}{(x-1)^3}$ .

- (ii) Describe the behaviour of the function  $y = \frac{(x+2)(x-1)^2}{x^3}$  near the horizontal asymptote.

(RF2.1, RF2.2, RF2.3)

### Resources/Notes

#### Authorized Resource

*Single Variable Essential Calculus  
Second Edition*

#### 1.6: Limits Involving Infinity

Student Book (SB): pp. 56 - 69

Instructor's Guide (IG): pp. 51 - 59

Power Point Lectures (PPL):

SteEC\_01\_06.ppt

## Relations and Functions

### Outcomes

*Students will be expected to*

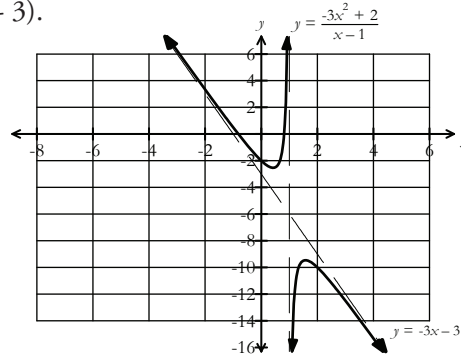
**RF2 Continued...**

### Achievement Indicators:

RF2.1, RF2.2, RF2.3 *Continued*

### Elaborations – Strategies for Learning and Teaching

Remind students that rational functions, such as  $y = \frac{-3x^2+2}{x-1}$ , can contain oblique (slant) asymptotes. These were introduced to students when they were working with limits. Students should be able to identify that an oblique asymptote exists since the degree of the numerator is exactly one more than the degree of the denominator. Remind them that the equation for the slant asymptote is the quotient from long division (i.e.,  $y = 3x - 3$ ).



Ask students to summarize their ideas concerning asymptotes. They should reference the following points:

- A rational function can have many vertical asymptotes.
- If a rational function has a horizontal asymptote, then it does not have an oblique one.
- The graph of a rational function can cross a horizontal asymptote, but does not cross a vertical asymptote.
- Horizontal/oblique asymptotes describe the end behaviour of the function; vertical asymptotes describe the behaviour of function near a point.

**General Outcome: Develop algebraic and graphical reasoning through the study of relations.**

### Suggested Assessment Strategies

#### *Paper and Pencil*

- Ask students to sketch the following graphs using intercepts, asymptotes, points of discontinuity, limits, sign diagram and test points.

$$(i) \quad y = \frac{x^3}{(x+1)^2}$$

$$(ii) \quad y = \frac{x^3+x^2-4x-4}{x^2+x-6}$$

$$(iii) \quad y = \frac{2x^3-7x^2-15x}{x^2-x-20}$$

(RF2.1, RF2.2, RF2.3)

### Resources/Notes

#### Authorized Resource

*Single Variable Essential Calculus  
Second Edition*

#### 3.4: Curve Sketching

SB: pp. 166 - 173

IG: pp. 175 - 184

PPL: SteEC\_03\_04.ppt

#### Note:

Questions #39 - 42 on p. 173 of the student book involve sketching rational functions with oblique asymptotes. Students should graph these functions using intercepts, points of discontinuity, limits, sign diagram and test points. They will sketch rational functions using derivatives later in this course.





# Derivatives

Suggested Time: 20 Hours

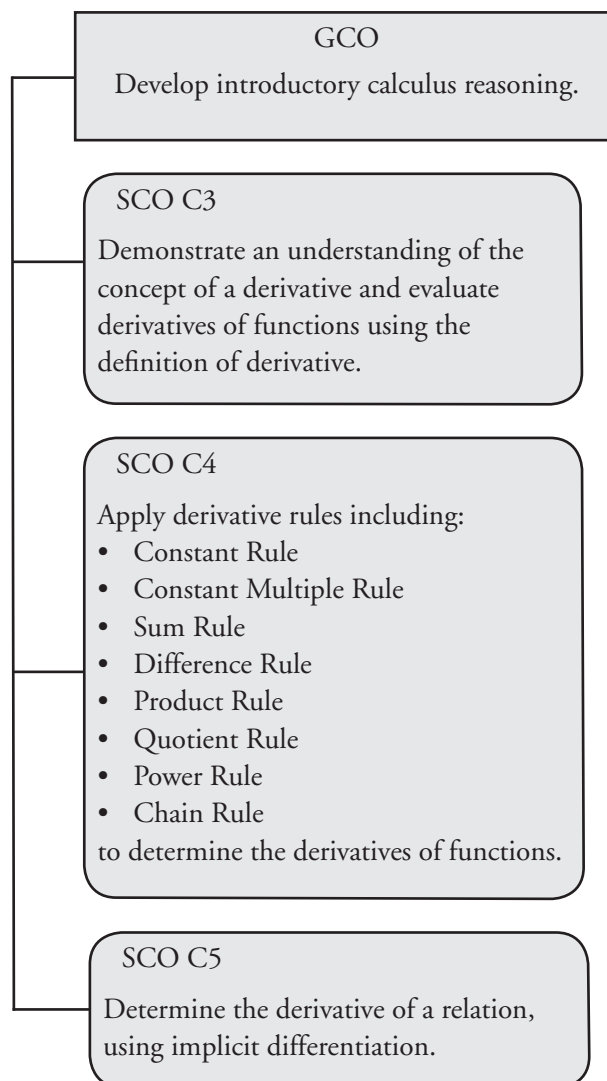
# Unit Overview

## Focus and Context

In this unit, students will differentiate functions using the definition of the derivative and will then apply the various rules of differentiation. Functions will be constructed from a sum, difference, product, quotient or composition of functions. They will also determine the derivative of a relation using implicit differentiation.

Students will determine the derivative of trigonometric functions, inverse trigonometric functions, exponential and logarithmic functions later in this course.

## Outcomes Framework



## SCO Continuum

Mathematics 2200	Mathematics 3200	Calculus 3208
	Relations and Functions	Calculus
not addressed	<p>RF11 Graph and analyze polynomial functions (limited to polynomial functions of degree <math>\leq 5</math>).</p> <p>[C, CN, T, V]</p>	<p>C3 Demonstrate an understanding of the concept of a derivative and evaluate derivatives of functions using the definition of derivative.</p> <p>[CN, ME, R, V]</p> <p>C4 Apply derivative rules including:</p> <ul style="list-style-type: none"> <li>• Constant Rule</li> <li>• Constant Multiple Rule</li> <li>• Sum Rule</li> <li>• Difference Rule</li> <li>• Product Rule</li> <li>• Quotient Rule</li> <li>• Power Rule</li> <li>• Chain Rule</li> </ul> <p>to determine the derivatives of functions.</p> <p>[C, CN, PS, R]</p> <p>C5 Determine the derivative of a relation, using implicit differentiation.</p> <p>[C, CN, PS, R, V]</p>

## Mathematical Processes

[C] Communication	[PS] Problem Solving
[CN] Connections	[R] Reasoning
[ME] Mental Mathematics and Estimation	[T] Technology
	[V] Visualization

## Calculus Reasoning

### Outcomes

*Students will be expected to*

**C3 Demonstrate an understanding of the concept of a derivative and evaluate derivatives of functions using the definition of derivative.**

[CN, ME, R, V]

### Achievement Indicators:

C3.1 Describe geometrically a secant line and a tangent line for the graph of a function at  $x = a$ .

C3.2 Determine the average rate of change of a function over an interval.

C3.3 Identify the instantaneous rate of change of a function at a point as the limiting value of a sequence of average rates of change.

### Elaborations—Strategies for Learning and Teaching

In Mathematics 1201, students connected slope to the rate of change of a line (RF3). In Science 1206, they represented the instantaneous velocity on a displacement-time graph using tangent lines. In this unit, introduce students to the concept of instantaneous rate of change and that the limiting value of the slopes of secant lines results in the slope of the tangent line.

In Mathematics 3200, students graphed and analyzed polynomial functions (limited to polynomial functions of degree less than or equal to 5) (RF11). Students should develop and use the definition of a derivative limited to polynomials of degree three or less, square root functions and rational functions with linear terms. They should investigate the differentiability of functions and be expected to draw derivative graphs given an original function and vice versa.

In Mathematics 1201, students calculated the slope of a line by choosing any two points on that line and dividing the difference in their  $y$ -values by the difference in their  $x$ -values (i.e., average rate of change). They discovered that any two points can be used in determining the slope of a line because the slope is constant throughout.

Provide students with the graph of a nonlinear function such as  $f(x) = x^3 - 2x^2 - 5x + 1$ . Ask them to describe where the slope of the function is positive, zero and negative. Students should observe there is no single slope for this graph since the function increases (slope is positive), levels off (slope is zero), and decreases (slope is negative) for various intervals of  $x$ .

Before students determine the slope of the function at a particular point (i.e., the slope of the tangent line to the curve at a particular point), they must first be able to distinguish between a tangent line and a secant line. Using graphing technology, the graph of the polynomial  $f(x) = x^3 - 2x^2 - 5x + 1$  could be illustrated on the same grid as the graphs of  $y = 2x + 5$ ,  $y = -x - 7$  and  $y = \frac{1}{2}x + 2$ . Ask students to identify which of these lines appear to touch the curve at one point.

The slope of a secant is the slope of the line joining two points on a curve. It is important for students to make the connection that the value of the slope of a secant line is the average rate of change of the function between two points  $P(a, f(a))$  and  $Q(b, f(b))$ .

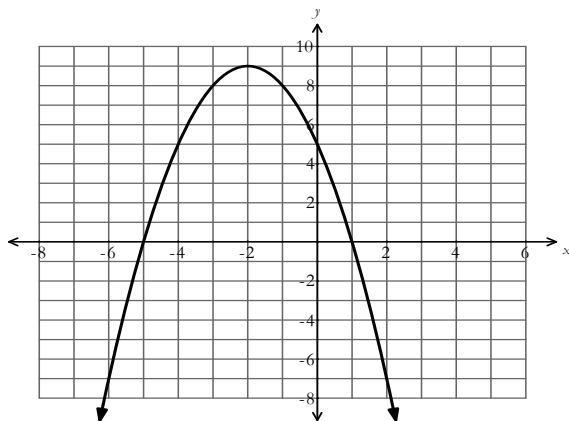
$$m_{avg} = \frac{f(b) - f(a)}{b - a}$$

**General Outcome: Develop introductory calculus reasoning.**

### Suggested Assessment Strategies

#### *Paper and Pencil*

- Given the graph shown, ask students to determine the average rate of change over the following intervals:
  - $x = -4$  to  $x = 1$
  - $x = -5$  to  $x = -1$
  - $x = -6$  to  $x = -3$



(C3.2)

### Resources/Notes

#### Authorized Resource

*Single Variable Essential Calculus  
Second Edition*

#### 2.1: Derivatives and Rates of Change

Student Book (SB): pp. 73 - 83  
Instructor's Guide (IG): pp. 71 - 78

Power Point Lecture (PPL):  
SteEC\_02\_01.ppt

Tools For Enriching Calculus Animations (TEC)

Visual 2.1A: Secant Line and Tangent

Visual 2.1B: Tangent Zoom

Video Examples: 1

## Calculus Reasoning

### Outcomes

Students will be expected to

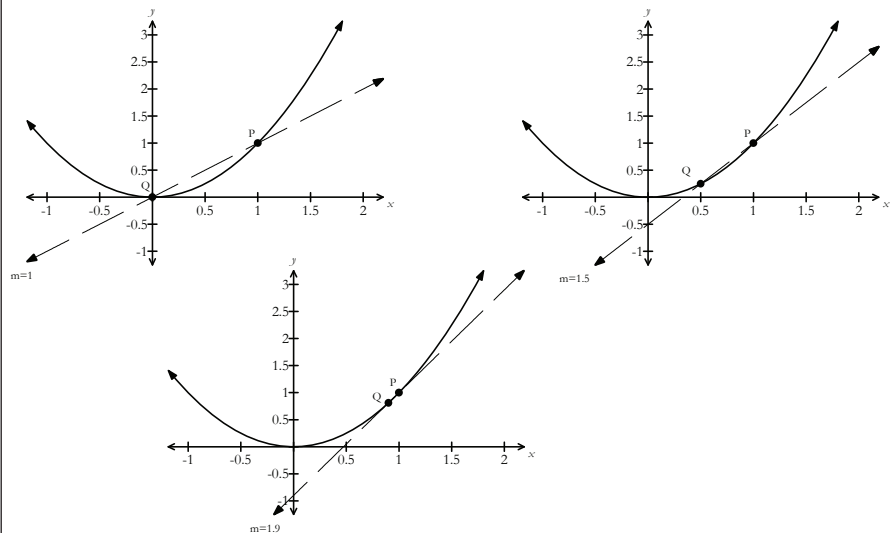
C3 Continued...

**Achievement Indicators:**

C3.1, C3.2, C3.3 Continued

### Elaborations – Strategies for Learning and Teaching

Ask students to predict what would happen to the slope of the secant line as point Q moves closer to point P. They could explore this using a function such as  $y = x^2$ . Ask students to find the slope of the tangent line at the point P(1, 1) by calculating the slopes of secant lines from P to Q as point Q moves closer to point P.



Ask students to calculate the slope and organize their results in a table.

Q(b, f(b)) approaching P(1, 1) from the left		
b	f(b)	$m_{avg} = \frac{f(b)-1}{b-1}$
0	0	1
0.5	0.25	1.5
0.9	0.81	1.9
0.99	0.9801	1.99
0.999	0.998001	1.999

Q(b, f(b)) approaching P(1, 1) from the right		
x	y	$m_{avg} = \frac{f(b)-1}{b-1}$
2	4	3
1.5	2.25	2.5
1.1	1.21	2.1
1.01	1.0201	2.01
1.001	1.002001	2.001

Students should notice that as Q approaches P from either side, the slope of the secant is approaching a particular value. This value is the slope of the tangent line. The slope of the tangent line (i.e., instantaneous rate of change) is the limiting value to the slope of the secant lines (i.e., average rate of change) as point Q moves closer to point P.

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**General Outcome: Develop introductory calculus reasoning.**


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**Suggested Assessment Strategies**
*Journal*

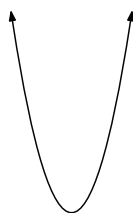
- Ask students to explain, using a diagram, why the slope of the tangent line to the curve  $f(x) = 2(x - 1)^2 + 3$  at the point (1,3) must be equal to 0.

(C3.1, C3.3)

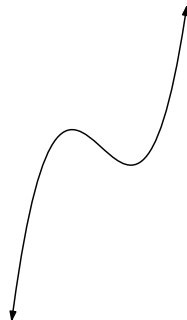
*Paper and Pencil*

- For each of the following curves, ask students to draw tangents that have negative slopes, positive slopes, and zero slopes.

(i)



(ii)



(C3.1)

- Ask students to predict the instantaneous rate of change of  $f(x) = x^2 + 2x - 1$  at  $x = 1$ . They should use tables of values to calculate a sequence of average rates of change.

(C3.3)

**Resources/Notes**
**Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**2.1: Derivatives and Rates of Change**

SB: pp. 73 - 83

IG: pp. 71 - 78

PPL: SteEC\_02\_01.ppt

TEC: Visual: 2.1A - Secant Line and Tangent, Visual 2.1B -

Tangent Zoom

Video Examples: 1

## Calculus Reasoning

### Outcomes

Students will be expected to

C3 Continued...

### Achievement Indicator:

C3.4 Define and evaluate the derivative at  $x = a$  as:

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

and

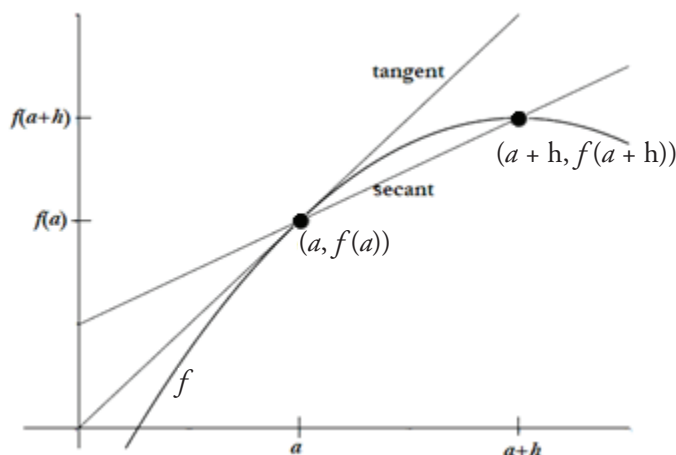
$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

### Elaborations – Strategies for Learning and Teaching

Symbolically, the limit of the slope of the secant lines is the slope of the tangent line as point Q approaches point P:

$$\lim_{Q \rightarrow P} m_{PQ} = m_{\text{tangent at P}}$$

Provide students with the following graph to help them develop the formula  $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ , where a new variable  $h$  is introduced.



As students observe the graph, they should notice the following:

- the variable  $h$  represents the distance between the  $x$ -coordinates  $a$  and  $a+h$ .
- the slope of the secant line to the curve  $f(x)$  is represented by:

$$m_{\text{secant}} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}$$

This is referred to as the difference quotient.

Ask students to predict what might happen if the distance between the two points on the curve decreases. They should notice that as  $h$  approaches zero, the slope of the secant is arbitrarily close to the slope of the tangent at the point  $(a, f(a))$ . The slope of the tangent to the curve  $f(x)$  at  $x = a$  is defined by:

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

Introduce this limit (provided it exists) as the derivative of the function  $f(x)$  at  $x = a$  written as:

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

This is the limit and difference quotient definition of the derivative.



**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Paper and Pencil*

- Ask students to determine the slope of the tangent line to the following curves at the specified  $x$ -value:

(i)  $f(x) = 7x - 4x^2$ ;  $x = 2$

(ii)  $f(x) = \sqrt{2x + 4}$ ;  $x = 6$

(iii)  $f(x) = \frac{2x + 1}{3x - 1}$ ;  $x = 1$

(C3.4)

**Resources/Notes****Authorized Resource***Single Variable Essential Calculus  
Second Edition***2.1: Derivatives and Rates of  
Change**

SB: pp. 73 - 83

IG: pp. 71 - 78

PPL: SteEC\_02\_01.ppt

## Calculus Reasoning

### Outcomes

Students will be expected to

C3 Continued...

### Achievement Indicator:

C3.4 Continued

### Elaborations – Strategies for Learning and Teaching

Ask students to use this limit to algebraically calculate the slope of the tangent line at  $x = 1$  for the function  $y = x^2$

$$f'(1) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$$

$$f'(1) = \lim_{h \rightarrow 0} \frac{(1+h)^2 - 1}{h}$$

Evaluating the limit, students should observe they have to simplify the expression since it results in an indeterminate form  $\frac{0}{0}$ . The resulting limit is 2, which implies that the slope of the tangent line is 2 (i.e.,  $f'(1) = 2$ )

An alternative form of determining the slope of the tangent line that touches the curve  $f(x)$ , at  $x = a$  is  $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$ . Ask students to use this limit to algebraically calculate the slope of the tangent line at  $x = 1$  for the function  $y = x^2$ . Using  $f(x) = x^2$ ,  $f(1) = 1$ , and the properties of limits should result in the following:

$$\lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1}$$

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$$

$$\lim_{x \rightarrow 1} \frac{(x-1)(x+1)}{x-1}$$

$$\lim_{x \rightarrow 1} (x+1)$$

$$2$$

The calculation of the limit above confirms that the slope of the tangent line at  $x = 1$  is 2. Finding the derivative function, without specifying a value for  $a$ , will be introduced next.

Students should interpret the derivative as the slope of a tangent. The tangent line to the curve  $y = f(x)$  at the point  $(a, f(a))$  is the line through  $(a, f(a))$  with slope  $f'(a)$ .

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**General Outcome: Develop introductory calculus reasoning.**

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**Suggested Assessment Strategies**

*Paper and Pencil*

- Each of the following represents the derivative of a function,  $f$ , at some number,  $a$ . Ask students to state  $f$  and  $a$  in each case.

(i) 
$$\lim_{x \rightarrow 0} \frac{(2+h)^2 - 2^2}{h}$$

(ii) 
$$\lim_{x \rightarrow 0} \frac{\sqrt{9+h} - 3}{h}$$

(C3.4)

**Resources/Notes**

**Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**2.1: Derivatives and Rates of  
Change**

SB: pp. 73 - 83

IG: pp. 71 - 78

PPL: SteEC\_02\_01.ppt

## Calculus Reasoning

### Outcomes

Students will be expected to

C3 Continued...

#### Achievement Indicators:

C3.5 Determine the equation of the tangent line and normal line to a graph of a relation at a given point.

C3.4 Continued

C3.6 Define and determine the derivative of a function using

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

(limited to polynomials of degree 3, square root and rational functions with linear terms).

### Elaborations – Strategies for Learning and Teaching

In Mathematics 1201, students used slope to determine whether two lines were parallel, perpendicular or neither (RF3). They also determined the equation of a linear function given the coordinates of a point on the line and the equation of a parallel or perpendicular line (RF7). Students should now determine the equation of the tangent and normal line to a curve at a given point. They can either use the point-slope form of a line or the slope-intercept form of a line. The previous example of finding the slope at  $x = 1$  to the curve  $y = x^2$  can be extended to show how to find the equation of the tangent line.

The concept of normal line is new to students. Teachers should define a normal line as a line that is perpendicular to the tangent line, and which intersects the curve at the point of tangency. Remind students that the slopes of perpendicular lines are negative reciprocals of each other.

Students previously observed that the slope of the tangent line is different at different points on the curve. The goal now is for students to write an expression that will give the slope for any  $x$ -value. The derivative of a function,  $f(x)$ , with respect to the variable,  $x$ , is the function  $f'(x)$  given by  $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ , provided the limit exists.

This limit definition for finding the derivative is also referred to as the method of First Principles.

Students should evaluate the derivative as a function of  $x$ , rather than merely the slope at a particular point. Ask students to calculate the derivative of a function such as  $y = x^2$ .

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{2xh + h^2}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} (2x + h)$$

$$f'(x) = 2x$$

As they use the limit properties, the slope of the curve  $y = x^2$  at any value of  $x$  is  $2x$ .

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**General Outcome: Develop introductory calculus reasoning.**


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**Suggested Assessment Strategies**
*Paper and Pencil*

- Ask students to answer the following:
    - (i) Find the equation of the tangent line and the normal line to the curve  $y = x^2$  at the point  $(-1, 1)$ .
    - (ii) Find the slope of the tangent to the curve  $f(x) = \frac{1}{x+1}$  at the point  $(0, 1)$ .
    - (iii) If  $f(x) = \frac{1}{x}$ , determine  $f'(3)$  and use it to find the equation of the tangent to the curve  $f(x) = \frac{1}{x}$  at the point  $(3, \frac{1}{3})$ .
- (C3.4, C3.5)
- Ask students to find the equation of the tangent line and the normal line to  $f(x) = \frac{1}{x+2}$  at  $x = -1$ .
- (C3.4, C3.5)
- Ask students to determine the derivative for each of the following functions using  $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ :
    - (i)  $f(x) = x^2 + 2$
    - (ii)  $f(x) = x^3 - x^2$
    - (iii)  $P(x) = \frac{1}{x+2}$
    - (iv)  $S(t) = \frac{1-2t}{3t+4}$
    - (v)  $f(x) = \sqrt{4+3x}$

(C3.6)

**Resources/Notes**
**Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**2.1: Derivatives and Rates of Change**

SB: pp. 73 - 83

IG: pp. 71 - 78

PPL: SteEC\_02\_01.ppt

Video Examples: 4 and 5

## Calculus Reasoning

### Outcomes

Students will be expected to

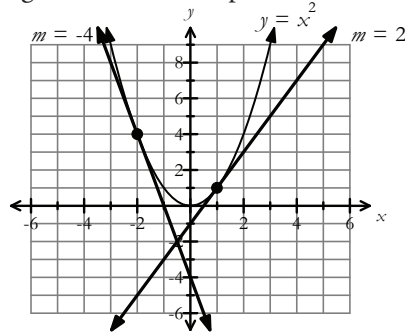
C3 Continued...

### Achievement Indicator:

C3.6 Continued

### Elaborations – Strategies for Learning and Teaching

Students can then determine the slope of the tangent at any point on the curve. Students should conclude the slope is 2 at  $x = 1$  and the slope is  $-4$  at  $x = -2$ . Ask students to sketch the parabola and plot the points  $(1, 1)$  and  $(-2, 4)$ . This visual should help them as they write the equation of the tangent lines at these points.



Students should determine the derivative for various functions limited to polynomials of degree 3, square root and rational functions with linear terms. The intent here is to have students work with a limited number of examples to understand the concept of the definition of the derivative prior to the development of differentiation rules.

Some common errors occur when students apply the definition of a derivative to a function such as  $f(x) = x^2 - 3$ . They incorrectly distribute the negative sign.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{(x+h)^2 - 3 - (x^2 - 3)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{(x^2 + 2xh + h^2 - 3) - (x^2 - 3)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - 3 - x^2 - 3}{h}$$

This error often leads them to incorrectly cancel the  $h$  term when simplifying the expression.

$$f'(x) = \lim_{h \rightarrow 0} \frac{2xh + h^2 - 6}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} 2x + h - 6$$

$$f'(x) = 2x - 6$$

---

**General Outcome: Develop introductory calculus reasoning.**


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**Suggested Assessment Strategies**
*Paper and Pencil*

- Ask students to find  $f'(x)$  for the following functions using

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

(i)  $f(x) = (x - 1)^2$

(ii)  $f(x) = \sqrt{x - 1}$

(C3.6)

*Performance*

- In groups of two, ask students to participate in the activity *Pass the Problem*. Distribute a problem to each pair that involves calculating a derivative using the limit and difference quotient definition. Ask one student to write the first line of the solution and then pass it to the second student. The second student will verify the workings and check for errors. If there is an error present, ask students to discuss the error and why it occurred. The student will then write the second line of the solution and pass it to their partner. This process continues until the solution is complete.

(C3.6)

**Resources/Notes**
**Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**2.1: Derivatives and Rates of Change**

SB: pp. 73 - 83

IG: pp. 71 - 78

PPL: SteEC\_02\_01.ppt

Video Examples: 4 and 5

## Calculus Reasoning

### Outcomes

Students will be expected to

C3 Continued...

#### Achievement Indicators:

C3.7 Use alternate notation interchangeably to express derivatives (i.e.,  $f'(x)$ ,  $\frac{dy}{dx}$ ,  $y'$  etc.).

C3.8 Determine whether a function is differentiable at a given point.

C3.9 Explain why a function is not differentiable at a given point, and distinguish between corners, cusps, discontinuities, and vertical tangents.

C3.10 Determine all values for which a function is differentiable, given the graph.

### Elaborations—Strategies for Learning and Teaching

Expose students to the different types of notation for the derivative. Point out to them that "prime" notations,  $f'(x)$  and  $y'$ , are from Lagrange and the  $\frac{dy}{dx}$  notations are from Leibniz.

$$f'(x) = y' = \underbrace{\frac{dy}{dx}}_{\text{Lagrange Notation}} = \underbrace{\frac{df}{dx}}_{\text{Leibniz Notation}} = \frac{d}{dx} f(x)$$

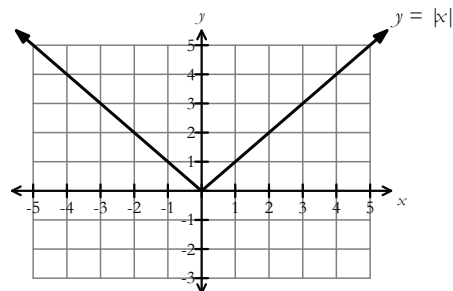
It is important for students to be aware that  $\frac{dy}{dx}$  is not a ratio, but rather another way of writing  $f'(x)$ . The advantage of the Leibniz notation is that both the independent variable and the dependent variable are shown in the derivative. When evaluating the derivative of  $y = f(x)$  at a certain value  $x = a$ , students can either write the notation as  $f'(a)$  or use Leibniz notation  $\left. \frac{dy}{dx} \right|_{x=a}$ .

Students have used the limit definition to find the derivative of a function and should realize that a function,  $f(x)$ , is differentiable at a point  $a$  if the limit exists. If it does not exist, then the function is not differentiable. Geometrically, differentiability of a function at  $a$  is saying that there is only one tangent line to the graph of the function at the point  $P(a, f(a))$  (i.e., the slopes of the secant lines approach a limit as  $x$  approaches  $a$ ).

Students should observe the graphs of functions that contain a corner point, cusp, vertical tangent or a discontinuity. The intent is that, given the graph of a function, students should be able to notice the values of  $x$  for which the function is not differentiable and to explain their reasoning (i.e., a corner point, a cusp, a vertical tangent or a discontinuity).

To investigate a function with a corner point, ask students to find  $f'(0)$  for a function such as  $f(x) = |x|$ . As students graph the function, they should notice a sharp corner at  $x = 0$  and might predict that the derivative does not exist.

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$



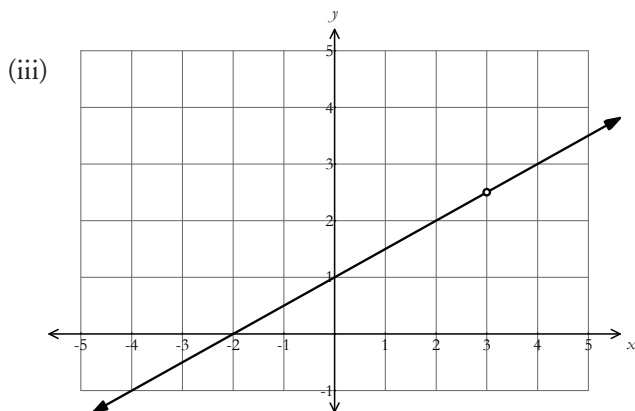
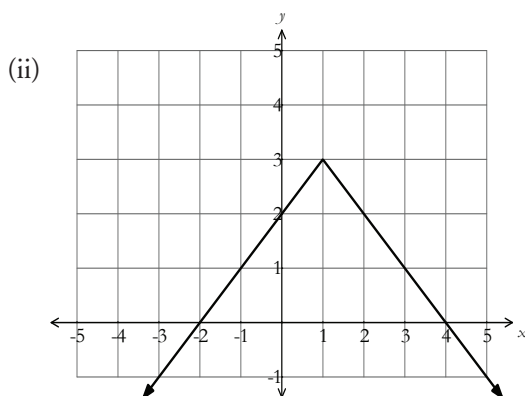
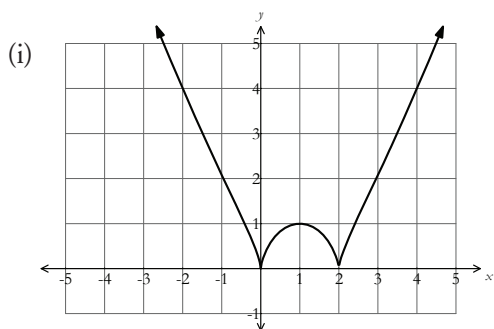


**General Outcome: Develop introductory calculus reasoning.**

### Suggested Assessment Strategies

*Paper and Pencil*

- Ask students to determine the value(s) of  $x$  for which the following functions are not differentiable. They should explain their reasoning.



(C3.8, C3.9, C3.10)

### Resources/Notes

#### Authorized Resource

*Single Variable Essential Calculus  
Second Edition*

#### 2.2: The Derivative as a Function

SB: pp. 84 - 95

IG: pp. 79 - 101

PPL: SteEC\_02\_02.ppt

Calculus Reasoning

Outcomes

Students will be expected to

C3 Continued...

Achievement Indicators:

C3.8, C3.9, C3.10 Continued

Elaborations – Strategies for Learning and Teaching

To verify this, ask students to investigate the derivative by evaluating one-sided limits.

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$$

$$f'(0) = \lim_{h \rightarrow 0} \frac{|0+h| - |0|}{h}$$

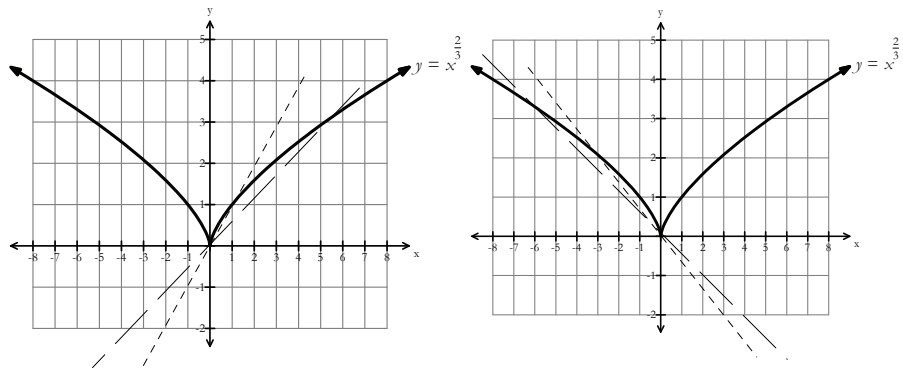
$$f'(0) = \lim_{h \rightarrow 0} \frac{|h|}{h}$$

Since the expression cannot be simplified, students should evaluate the two one-sided limits.

left-hand limit	right-hand limit
$\lim_{h \rightarrow 0^-} \frac{ h }{h}$	$\lim_{h \rightarrow 0^+} \frac{ h }{h}$
$\lim_{h \rightarrow 0^-} \frac{-h}{h}$	$\lim_{h \rightarrow 0^+} \frac{h}{h}$
$\lim_{h \rightarrow 0^-} -1$	$\lim_{h \rightarrow 0^+} 1$
-1	1

Students should recall from the previous unit that since the two one-sided limits are different, the limit does not exist. As a result, the derivative  $f'(0)$  does not exist either. They should also observe there is no unique tangent at this point.

A cusp is an extreme case of a corner point where the slopes of the secant lines approach  $\infty$  from one side of the curve and  $-\infty$  from the other. Provide students with the graph of a function such as  $f(x) = x^{\frac{2}{3}}$ . They should investigate the right and left-hand limits at the point (0,0) by drawing a sequence of secant lines.



Ask students what they notice as the slope of the secant lines becomes steeper and steeper. They should comment that the slope of the secant line is approaching  $\infty$  as  $h$  approaches  $0^+$ . Similarly, slope of the secant line is approaching  $-\infty$  as  $h$  approaches  $0^-$ . Therefore the limit does not exist. From these conclusions, students should state that the graph is not differentiable at  $x = 0$ .

## General Outcome: Develop introductory calculus reasoning.

### Suggested Assessment Strategies

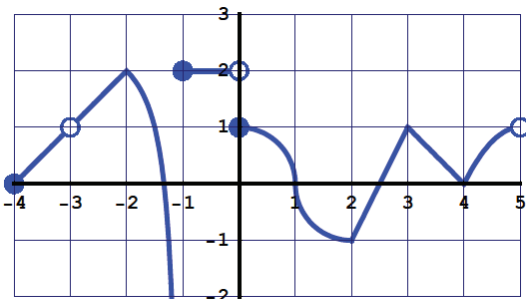
#### Performance

- Ask students to play *Four Corners* in which each student is given the graph of a function that is not differentiable at a point. Each corner of the classroom would be designated as either “corner”, “cusp”, “vertical tangent” or “discontinuity”. The students would then proceed to the corner of the room that corresponds to the reason their function is not differentiable. They should explain to the members of their group why their function is not differentiable.

(C3.8, C3.9, C3.10)

#### Paper and Pencil

- Ask students to find all values of  $x$  for which the function is not differentiable. Determine why the function is not differentiable at these values of  $x$ .



(C3.8, C3.9, C3.10)

### Resources/Notes

#### Authorized Resource

*Single Variable Essential Calculus  
Second Edition*

#### 2.2: The Derivative as a Function

SB: pp. 84 - 95

IG: pp. 79 - 101

PPL: SteEC\_02\_02.ppt

## Calculus Reasoning

### Outcomes

Students will be expected to

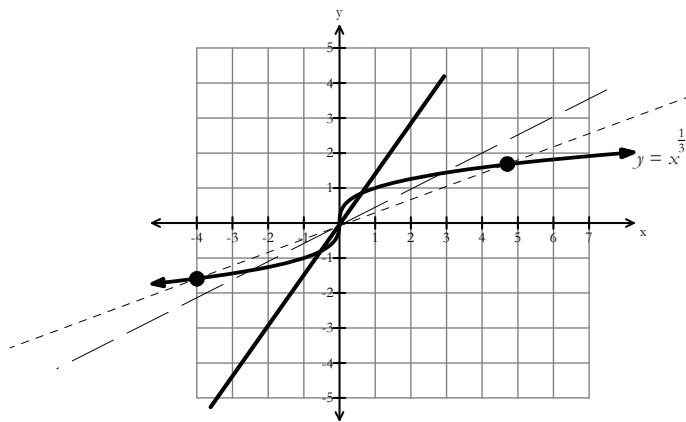
C3 Continued...

### Achievement Indicators:

C3.8, C3.9, C3.10 *Continued*

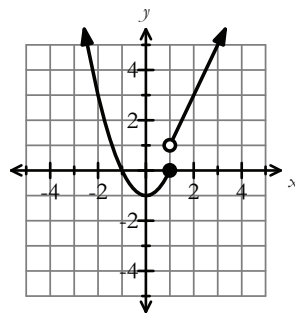
### Elaborations – Strategies for Learning and Teaching

A vertical tangent exists when the slopes of the secant lines approach either  $\infty$  or  $-\infty$  from both sides of the curve. Teachers should provide students with the graph of a function such as  $f(x) = x^{\frac{1}{3}}$ . To determine if the derivative at the point  $(0,0)$  exists, ask them to draw a sequence of secant lines.



Students should observe the secant lines begin to take on the characteristic of a vertical line. The slope of the secant line is approaching  $\infty$  from both sides of  $x = 0$ , therefore, the derivative does not exist at  $x = 0$ . Ask students to verify, given the graph of the function  $f(x) = \sqrt[3]{2-x}$ , that the function has a vertical tangent at  $x = 2$  by showing the slope of the secant line is approaching  $-\infty$  from both sides of  $x = 2$ .

A function that has a point of discontinuity is another example where the derivative at a point does not exist. Observing the graph below, students should notice there is point of discontinuity at  $x = 1$ .



Reinforce that if a function is not continuous at  $x = a$  then it is not differentiable at  $x = a$ . Students should notice the function  $f(x)$  is not continuous at  $x = 1$ , therefore, it is not differentiable at  $x = 1$ . This is a good time to emphasize that while differentiability implies continuity, the converse is not necessarily true.

---

**General Outcome: Develop introductory calculus reasoning.**


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**Suggested Assessment Strategies**
*Paper and Pencil*

- Ask students to show that the function is not differentiable at the indicated number and provide a geometrical explanation.

(i)  $y = |x - 5|$ ,  $a = 5$

(ii)  $y = \sqrt[3]{x + 2}$ ,  $a = -2$

(C3.8, C3.9, C3.10)

- Provide students with the graph of the following piecewise function.

$$y = \begin{cases} x^2, & -5 \leq x < 1 \\ (x - 2)^2, & 1 \leq x \leq 7 \end{cases}$$

Ask them to answer the following:

- Is the function continuous at  $x = 1$ ? Explain your reasoning.
- Is the function differentiable at  $x = 1$ ? Is there a cusp, vertical tangent, discontinuity or corner point on the graph at  $x = 1$ ? How do you know?

(C3.8, C3.9, C3.10)

**Resources/Notes**
**Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**2.2: The Derivative as a Function**

SB: pp. 84 - 95

IG: pp. 79 - 101

PPL: SteEC\_02\_02.ppt

## Calculus Reasoning

### Outcomes

Students will be expected to

C3 Continued...

#### Achievement Indicators:

C3.11 Sketch a graph of the derivative of a function, given the graph of a function.

C3.12 Sketch a graph of the function, given the graph of the derivative of a function.

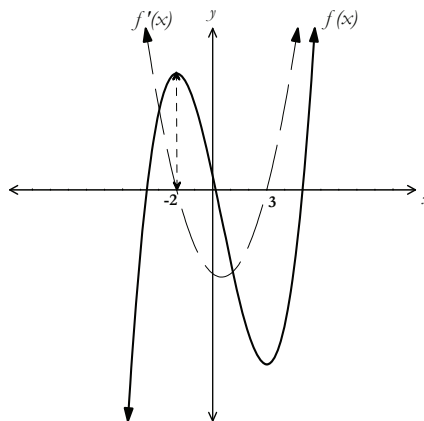
### Elaborations—Strategies for Learning and Teaching

When students construct the graph of  $f'(x)$  given the graph of  $f(x)$ , they should think about the following points:

- When the graph of  $f(x)$  has a positive slope, the graph of  $f'(x)$  will be above the  $x$ -axis.
- When the graph of  $f(x)$  has zero slope the graph of  $f'(x)$  will cross the  $x$ -axis.
- When the graph of  $f(x)$  has a negative slope, the graph of  $f'(x)$  will be below the  $x$ -axis.

Using the definition of the derivative for a polynomial function, students should have noticed that the degree of the derivative is always one less than the degree of the original polynomial. Graphically, this means there will be one less turning point. When asking students to sketch the original function from a derivative graph, avoid a scale on the vertical axis for the original function.

To illustrate the points made above, the following graph could be used.



Students should observe the following:

- The  $x$ -coordinates of the local maxima/minima of  $f(x)$  correspond to the  $x$ -intercepts on the graph of  $f'(x)$ .
- When the graph of  $f(x)$  is increasing the graph of  $f'(x)$  lies above the  $x$ -axis and when the graph of  $f(x)$  is decreasing the graph of  $f'(x)$  lies below the  $x$ -axis.

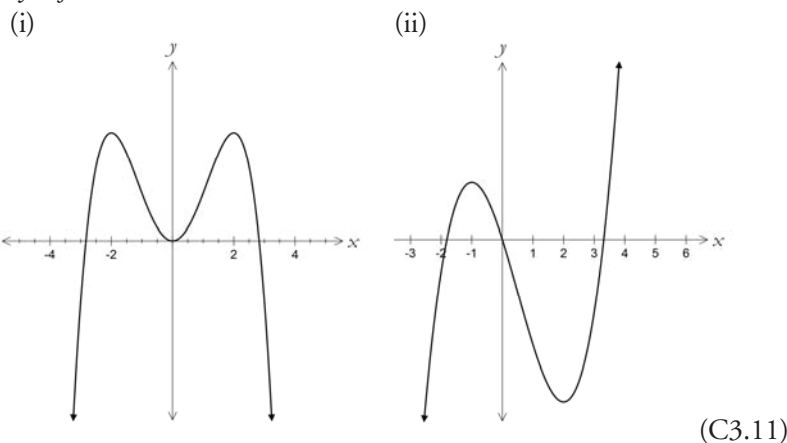
**General Outcome: Develop introductory calculus reasoning.**

**Suggested Assessment Strategies**

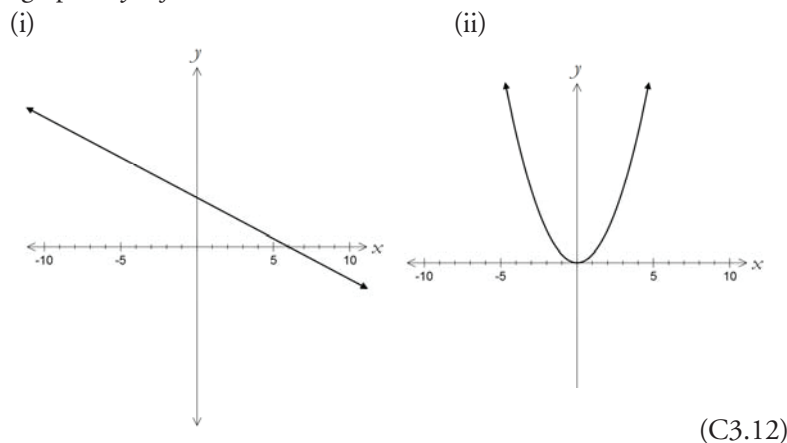
**Resources/Notes**

*Paper and Pencil*

- Ask students to sketch the graph of  $y = f'(x)$  given the graph of  $y = f(x)$ .

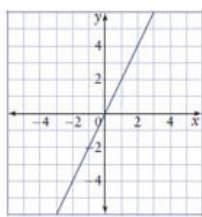
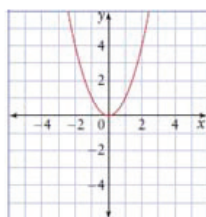


- Ask students to sketch the graph of a possible  $y = f(x)$  given the graph of  $y = f'(x)$ .



- For the activity *Find Your Partner*, each student should be given a card containing the graph of  $f(x)$  or  $f'(x)$ . Ask students to move around the classroom to match their graph with either the corresponding  $f'(x)$  or  $f(x)$  graph (i.e., find the partner who is your derivative or find the partner of whom you are the derivative). Encourage them to justify their choices.

Sample Graphs:  $f(x)$   $f'(x)$



(C3.11, C3.12)

**Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**2.2: The Derivative as a Function**

SB: pp. 84 - 95

IG: pp. 79 - 101

PPL: SteEC\_02\_02.ppt

TEC: Visual 2.2: Slope-a-Scope

Video Examples: 1 and 2

## Calculus Reasoning

### Outcomes

*Students will be expected to*

**C4 Apply derivative rules including:**

- Constant Rule
- Constant Multiple Rule
- Sum Rule
- Difference Rule
- Product Rule
- Quotient Rule
- Power Rule
- Chain Rule

to determine the derivatives of functions.

[C, CN, PS, R]

### Achievement Indicator:

*C4.1 Derive the Constant, Constant Multiple, Sum, Difference, Product, and Quotient Rules for determining derivatives.*

### Elaborations—Strategies for Learning and Teaching

Students are introduced to the derivative rules to determine the derivative of functions. A function is constructed from a sum, difference, product, quotient or composition of functions. Derivatives of trigonometric functions, exponential and logarithmic functions will be introduced later in this course. It is important for students to understand the rules, why they work, and how they work. The derivations of the Power Rule and the Chain Rule are very complex and it is not the intent of this course that students be exposed to these proofs.

The derivation of these rules provides a great opportunity for group discussion. It allows students to see how the definition of a derivative can be used to prove that these rules are true. Teachers should derive the Constant Rule, Sum and Difference Rules, Product and Quotient Rules, with students. To this point, students have had little or no exposure to proofs. Teachers should lead a discussion on the basic methods of proofs, as well as the importance of proofs in mathematics.

#### • Constant Rule

Teachers should use a graph and the definition of the derivative when introducing the Constant Rule. When graphing a horizontal line, such as  $f(x) = 3$ , students should observe that, at any point, the tangent line is the same as the horizontal line which has a slope of zero. It then follows that the derivative at any point must also be zero. Applying the definition of the derivative, verifies that if  $f(x) = c$ , where  $c$  is a constant, then  $f'(x) = 0$  or  $\frac{d}{dx}(c) = 0$ .

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{c - c}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{0}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} 0$$

$$f'(x) = 0$$



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**General Outcome: Develop introductory calculus reasoning.**

---

**Suggested Assessment Strategies**

*Paper and Pencil*

- Ask students to use the definition of the derivative to prove the:  
Constant Rule.

(C4.1)

**Resources/Notes**

**Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**2.3: Basic Differentiation  
Formulas**

SB: pp. 95 - 107

IG: pp. 102 - 108

PPL: SteEC\_02\_03.ppt

## Calculus Reasoning

## Outcomes

Students will be expected to

C4 Continued...

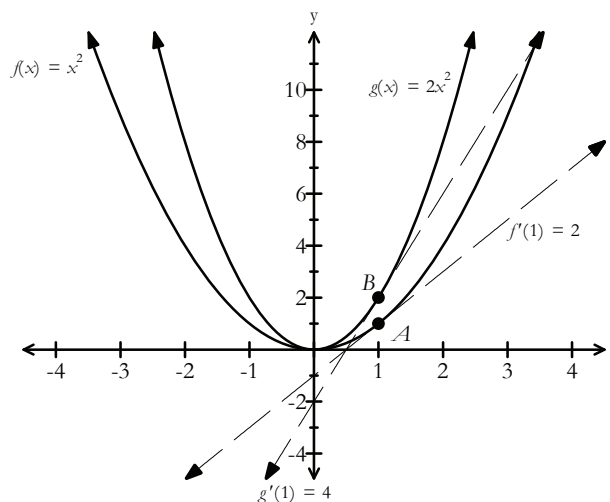
## Achievement Indicator:

C4.1 Continued

## Elaborations—Strategies for Learning and Teaching

- Constant Multiple Rule

Similarly, the derivative of a constant times a function is equivalent to the constant times the derivative of the function. Ask students to consider a function  $f(x)$  and a constant  $c = 2$ . They can create a new function  $g(x)$  by multiplying  $f(x)$  by the constant  $c$ . Using the visual representation below, students should observe that the curve  $g(x)$  is displaced vertically from  $f(x)$  by a factor of two.



Ask students to think about how the tangent lines to the two curves at the same  $x$ -value differ. Students are aware that the derivatives are the slopes of the tangent lines to the curves at each value of  $x$ . They should observe the slope of the tangent line to  $g(x)$  is twice that of the tangent line to  $f(x)$ . To verify the constant multiple rule, ask students to apply the definition of the derivative. If  $g(x) = c f(x)$  then  $g'(x) = c f'(x)$  where  $c$  is a constant.

$$g'(x) = \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h}$$

$$g'(x) = \lim_{h \rightarrow 0} \frac{c f(x+h) - c f(x)}{h}$$

$$g'(x) = c \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$g'(x) = c f'(x)$$

Remind students that, in order to bring the constant  $c$  outside the limit, they are applying the limit law which states that the limit of a constant times a function is the constant times the limit of the function.

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**General Outcome: Develop introductory calculus reasoning.**

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**Suggested Assessment Strategies***Paper and Pencil*

- Ask students to use the definition of the derivative to prove the:  
Constant Multiple Rule.

(C4.1)

**Resources/Notes****Authorized Resource***Single Variable Essential Calculus  
Second Edition***2.3: Basic Differentiation  
Formulas**

SB: pp. 95 - 107

IG: pp. 102 - 108

PPL: SteEC\_02\_03.ppt

## Calculus Reasoning

### Outcomes

Students will be expected to

C4 Continued...

### Achievement Indicator:

C4.1 Continued

### Elaborations – Strategies for Learning and Teaching

- **Sum and Difference Rules**

Ask students to continue to use the definition of the derivative to prove the derivative of the sum/difference of functions. If  $k(x) = f(x) - g(x)$  then  $k'(x) = f'(x) - g'(x)$ .

$$k'(x) = \lim_{h \rightarrow 0} \frac{k(x+h) - k(x)}{h}$$

$$k'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - g(x+h) - (f(x) - g(x))}{h}$$

$$k'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - g(x+h) - f(x) + g(x)}{h}$$

$$k'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x) - g(x+h) + g(x)}{h}$$

$$k'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} - \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h}$$

$$k'(x) = f'(x) - g'(x)$$

This can also be written as  $\frac{d}{dx}(f - g) = \frac{d}{dx}(f) - \frac{d}{dx}(g)$ . Similarly, students should notice that the derivative of a sum of two functions is equal to the sum of the derivatives of each function.

- **Product and Quotient Rules**

Previously, students recognized that when they added two functions together, the derivative of the sum is the sum of the derivatives. They may initially conclude, therefore, that if they multiply two functions together, then the derivative of the product is the product of the derivatives. If  $f(x)$  and  $g(x)$  are differentiable then the product  $p(x) = f(x) \times g(x)$  is differentiable. Ask students to use the definition of the derivative to prove  $p'(x) = f'(x)g(x) + f(x)g'(x)$ .

Similarly, if two functions  $f(x)$  and  $g(x)$  are differentiable, then the quotient is differentiable. The proof of the Quotient Rule can be illustrated using the definition of a derivative. As an alternative method, encourage students to prove the Quotient Rule using the Product Rule.

---

**General Outcome: Develop introductory calculus reasoning.**


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**Suggested Assessment Strategies**
*Paper and Pencil*

- Ask students to use the definition of the derivative to prove the:
  - (i) Sum Rule
  - (ii) Difference Rule
  - (iii) Product Rule
  - (iv) Quotient Rule

(C4.1)

*Portfolio*

- Ask students to collect and retain their proofs. As the course continues, students can add to their portfolios other proofs (e.g.,  $\frac{d}{dx} e^x = e^x$ ,  $\frac{d}{dx} \sin x = \cos x$ , etc.)

(C4.1)

**Resources/Notes**
**Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**2.3: Basic Differentiation**
**Formulas**

SB: pp. 95 - 107

IG: pp. 102 - 108

PPL: SteEC\_02\_03.ppt

**2.4: The Product and Quotient**
**Rules**

SB: pp. 107 - 114

IG: pp. 109 - 115

PPL: SteEC\_02\_04.ppt

**Suggested Resource**

Resource Link:

[www.k12pl.nl.ca/curr/10-12/math/3208/res-link/deriv.html](http://www.k12pl.nl.ca/curr/10-12/math/3208/res-link/deriv.html)

- formal proof of the product rule
- interactive exploration of the sum and difference of functions

## Calculus Reasoning

### Outcomes

Students will be expected to

C4 Continued...

#### Achievement Indicators:

C4.1 Continued

C4.2 Determine derivatives of functions, using the Constant, Constant Multiple, Power, Sum, Difference, Product and Quotient Rules.

### Elaborations – Strategies for Learning and Teaching

If  $Q(x) = \frac{f(x)}{g(x)}$ , where  $g(x) \neq 0$ , students can rewrite the equation as  $f(x) = Q(x) \times g(x)$  and apply the Product Rule,  $f'(x) = Q'(x)g(x) + Q(x)g'(x)$ . Ask students to solve for  $Q'(x)$ .

$$Q'(x)g(x) = f'(x) - Q(x)g'(x)$$

$$Q'(x)g(x) = f'(x) - \frac{f(x)}{g(x)}g'(x)$$

$$Q'(x)g(x) = f'(x)\frac{g(x)}{g(x)} - \frac{f(x)}{g(x)}g'(x)$$

$$Q'(x)g(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)}$$

$$Q'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

Students should practice applying the Product Rule and the Quotient Rule in appropriate situations.

It is important to note to students that the Power Rule applies only to power functions  $y = x^n$ . It does not apply to exponential functions such as  $y = 2^x$ . They will study the derivatives of exponential functions later in this course.

The rule states that if  $n$  is any real number then  $\frac{d}{dx}(x^n) = nx^{n-1}$ . The Power Rule is valid for all exponents, whether negative, fractional, or irrational. Ask students to determine the derivative of a function such as  $f(x) = x^7$  using the Power Rule (i.e.,  $f'(x) = 7x^6$ ). Alternatively, they could write this function as a product such as  $f(x) = x^4 \cdot x^3$ . Applying the product rule,  $f'(x) = 4x^3(x^3) + (x^4)(3x^2)$  simplifies to  $f'(x) = 7x^6$ . When finding the derivative of  $f(x) = x^4 \cdot x^3$ , students may mistakenly write  $f'(x) = \frac{d}{dx}(x^4) \cdot \frac{d}{dx}(x^3)$  resulting in  $f'(x) = 4x^3 \cdot 3x^2$ . When they then multiply the derivatives, the result is  $12x^5$ . This example reinforces that the derivative of the product is not the product of the derivatives.

**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Paper and Pencil*

- Ask students to determine  $f'(x)$  for each of the following functions:

(i)  $f(x) = 4x^2 - 3x^3 + 5$

(ii)  $f(x) = \sqrt{x} + x^3$

(iii)  $f(x) = x^3\sqrt{x}$

(iv)  $f(x) = \sqrt{x} + \frac{2}{\sqrt{x}} + \frac{3}{\sqrt{\pi+3}}$

(C4.2)

- Ask students to differentiate the following:

(i)  $y = \frac{x+2}{x+1}$

(ii)  $y = \frac{3x+1}{2x-5}$

(iii)  $y = \frac{2x^2-x}{x^2+1}$

(iv)  $y = \frac{6-\sqrt{x}}{1+\sqrt{x}}$

(C4.2)

- Ask students to evaluate  $(f \times g)'(2)$  given the following information:

$f(2) = 5, f'(2) = 6, g(2) = 7$  and  $g'(2) = -1$ .

(C4.2)

- Ask students to evaluate  $\left(\frac{f}{g}\right)'(2)$  given the following information:

$f(2) = 3, f'(2) = 5, g(2) = -1$  and  $g'(2) = -4$ .

(C4.2)

- Ask students to differentiate  $P(x) = (3x+2)(4x-3)$  by expanding the function. They should then verify their answer using the Product Rule. Ask them to discuss which method they prefer and why.

(C4.2)

*Interview*

- With reference to the above question, ask students to answer the following questions:

(i) Which method is more efficient?

(ii) Will this apply in all cases? Explain your reasoning.

(iii) Using their preferred method, ask students to differentiate

$P(x) = (4x^3 + 2x^2 + 5x)(-x^2 - 3x + 11)$ . Ask students if they still agree with their choice.

(C4.2)

**Resources/Notes****Authorized Resource***Single Variable Essential Calculus  
Second Edition***2.3: Basic Differentiation  
Formulas**

SB: pp. 95 - 107

IG: pp. 102 - 108

PPL: SteEC\_02\_03.ppt

**2.4: The Product and Quotient  
Rules**

SB: pp. 107 - 114

IG: pp. 109 - 115

PPL: SteEC\_02\_04.ppt

## Calculus Reasoning

### Outcomes

Students will be expected to

C4 Continued...

#### Achievement Indicators:

C4.3 Determine second and higher-order derivatives of functions.

C4.4 Determine derivatives of functions using the Chain Rule.

### Elaborations—Strategies for Learning and Teaching

The differentiation process should be continued to find the second, third, and successive derivatives of  $f(x)$ , which are called higher order derivatives of  $f(x)$ .

The second derivative is found by taking the derivative of the first derivative and is written as  $y'' = f''(x) = \frac{d^2y}{dx^2}$ . The third derivative would be the derivative of the second derivative and is written as  $y''' = f'''(x) = \frac{d^3y}{dx^3}$ .

The multiple prime notation will eventually become cumbersome for higher derivatives. The following notation can be used to denote the  $n^{\text{th}}$  derivative of  $f(x)$ ,  $y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n}$ .

Inform students that the superscripts used for higher-order derivatives, such as  $y^{(4)}$  for the fourth derivative of  $y$ , are not exponents. The presence of the parenthesis in the exponent denotes differentiation (i.e.,  $f^{(2)}(x) = f''(x)$ ) while the absence of the parenthesis denotes exponentiation (i.e.,  $f^2(x) = [f(x)]^2$ ).

First and second derivatives will be discussed in more detail in this course when students are exposed to the applications of derivatives such as curve sketching, velocity and acceleration.

The chain rule exists for differentiating composite functions. A good way to introduce the chain rule is to have students explore patterns when finding the derivative of a series of functions such as  $f(x) = (x - 2)^2$  and  $g(x) = (x + 3)^3$ . It may be easier to see the pattern if students write the derivative in factored form (i.e.,  $f'(x) = 2(x - 2)$  and  $g'(x) = 3(x + 3)^2$ ). Students may initially conclude that the derivative is simply the power rule applied to the binomial. It is important for them to realize this is not always the case. To provide a counterexample, ask students to find the derivative of  $y = (x^2 + 1)^3$  and see if the pattern holds true. Using the pattern, they may incorrectly predict the answer to be  $y' = 3(x^2 + 1)^2$ . With teacher guidance, ask students to expand the binomial and differentiate using the power rule on each term.

$$y = (x^2 + 1)^3$$

$$y = x^6 + 3x^4 + 3x^2 + 1$$

$$y' = 6x^5 + 12x^3 + 6x$$

$$y' = 6x(x^4 + 2x^2 + 1)$$

$$y' = 6x(x^2 + 1)^2$$

$$y' = 3(2x)(x^2 + 1)^2$$

Ask students why the term  $6x$  is written as  $3(2x)$ . They should notice that  $2x$  is the derivative of  $x^2 + 1$ , the inner function.



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**General Outcome: Develop introductory calculus reasoning.**


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**Suggested Assessment Strategies**
*Paper and Pencil*

- Ask students to answer the following:
  - (i) Determine the first four derivatives of  $f(x) = -2x^3 + 5x^2 + 12$ .
  - (ii) Given  $f(x) = x^5 + 3x^3 + 42$ , determine the value of  $n$ , such that  $f^{(n)}(x) = 0$ .
  - (iii) Determine the point on the graph of  $f(x) = -x^3 + 12x^2 + 4x - 10$  where  $f''(x) = 0$ .

(C4.3)

**Resources/Notes**
**Authorized Resource**
*Single Variable Essential Calculus  
Second Edition*
**2.4: The Product and Quotient Rules**

SB: pp. 107 - 114

IG: pp. 109 - 115

PPL: SteEC\_02\_04.ppt

**2.2: The Derivative as a Function**

SB: pp. 84 - 95

IG: pp. 79 - 101

PPL: SteEC\_02\_02.ppt

## Calculus Reasoning

### Outcomes

Students will be expected to

C4 Continued...

**Achievement Indicator:**

C4.4 Continued

### Elaborations—Strategies for Learning and Teaching

Ask students to rewrite  $y = (x^2 + 1)^3$  as a composition of functions  $f(g(x))$ . Ensure they define the outer function as  $f(x) = x^3$  and the inner function  $g(x) = x^2 + 1$  (i.e.,  $f(g(x)) = f(x^2 + 1) = (x^2 + 1)^3$ ). They already determined the derivative to be  $\frac{d}{dx} f(g(x)) = 3(x^2 + 1)^2(2x)$ . Ask students how the values  $3(x^2 + 1)^2$  and  $(2x)$  are related to  $f'(g(x))$  and  $g'(x)$ .

The Chain Rule is a way of finding the derivative of a composition of functions. If  $C(x) = (f \circ g)(x) = f(g(x))$ , then

$$C'(x) = \underbrace{f'(g(x))}_{\substack{\text{derivative of} \\ \text{the outer} \\ \text{function with} \\ \text{respect to the} \\ \text{inner function}}} \cdot \underbrace{g'(x)}_{\substack{\text{derivative of the} \\ \text{inner function}}}$$

Alternatively, if  $y$  is a function of  $u$  and  $u$  is a function of  $x$ , then the derivative of  $y$  with respect to  $x$  is:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad (\text{Leibniz notation for the Chain Rule})$$

This version of the chain rule can also be applied when looking at derivatives as rates of change. Teachers can use an example such as the following:

A car travels twice as fast as a bicycle.

A bicycle travels 4 times as fast as a person walking.

A car travels ? times as fast as a person walking.

Students should be able to make the connection that the first rate is the product of the other two (i.e., car is travelling 8 times as fast as the person walking).

This can be applied to any situation involving related instantaneous rates of change. If  $y$  changes  $a$  times as fast as  $u$  ( $a = \frac{dy}{du}$ ), and  $u$  changes  $b$  times as fast as  $x$  ( $b = \frac{du}{dx}$ ), then  $y$  changes  $ab$  times as fast as  $x$ . This can be written as  $\frac{dy}{dx} = a \cdot b = \frac{dy}{du} \cdot \frac{du}{dx}$ .

**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Paper and Pencil*

- Ask students to determine the following:

$$(i) \quad \frac{d}{dx}(x+1)^2 \qquad (ii) \quad \frac{d}{dx}(2x+1)^3$$

$$(iii) \quad \frac{d}{dx}(x^2+1)^{50} \qquad (iv) \quad \frac{d}{dx}\sqrt{x^2+x}$$

(C4.4)

- Ask students to determine  $\frac{dy}{dx}$  if  $y = u^{10} - 7u^3 + 1$  where  $u = 3x^2 - 2x$ .

(C4.4)

- Given  $h(x) = f(g(x))$  where  $g(2) = 3$ ,  $g'(2) = 5$ ,  $f'(2) = 5$ ,  $f'(3) = -2$  and  $f'(5) = 4$ , ask students to evaluate  $h'(2)$ .

(C4.4)

- Ask students to differentiate  $f(x) = (3x^2 + 1)^2$  by expanding the function. They should then verify their answer using the Chain rule.

(C4.4)

**Resources/Notes****Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**2.5: The Chain Rule**

SB: pp. 114 - 122

IG: pp. 116 - 123

PPL: SteEC\_02\_05.ppt

Video: Example 4

## Calculus Reasoning

### Outcomes

*Students will be expected to*

**C4 Continued...**

**Achievement Indicators:**

C4.4 *Continued*

C4.5 *Solve problems involving derivatives drawn from a variety of applications, limited to tangent and normal lines, straight line motion and rates of change.*

### Elaborations—Strategies for Learning and Teaching

Referencing the previous example,  $y = (x^2 + 1)^3$ , ask students to demonstrate that the Leibniz notation for the Chain rule leads to the same result. Guide them through the process by re-writing  $y = (u)^3$  where  $y$  is a function of  $u$  and  $u = x^2 + 1$  where  $u$  is a function of  $x$ . Ask them to differentiate where  $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ .

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{d}{du}(u^3) \cdot \frac{d}{dx}(x^2 + 1)$$

$$\frac{dy}{dx} = 3u^2 \cdot 2x$$

$$\frac{dy}{dx} = 3(x^2 + 1)^2 \cdot 2x$$

Later in this course, students will apply their knowledge of the first derivative to find absolute and relative extrema of a continuous function on a closed interval.

There are many applications of derivatives including tangent and normal lines, rates of change, straight line motion, optimization and related rates. Optimization and related rates will be dealt with in the next unit.

Students should be aware that the concept of derivative can be interpreted as a rate of change. They should use derivatives to compute:

- velocity (rate of change of displacement with respect to time)
- acceleration (rate of change of velocity with respect to time)
- rate of change of population size with respect to time
- equation of a tangent and/or normal line
- other rates of change in physics, biology and/or chemistry

If an object travels along a horizontal line, (e.g., the  $x$ -axis) and displacement  $s(t)$  is measured over time, then the first derivative  $s'(t)$ , equals the velocity  $v(t)$  of that object. The second derivative  $s''(t) = v'(t)$  equals the acceleration  $a(t)$  of that object.

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**General Outcome: Develop introductory calculus reasoning.**


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**Suggested Assessment Strategies**
*Paper and Pencil*

- Ask students to differentiate the following functions:

(a)  $f(x) = (2x + 1)(4x - 1)^5$

(b)  $y = \frac{(2x + 3)^3}{\sqrt{4x - 7}}$

(C4.4)

- Ask students to determine the value of  $\frac{dy}{dx}$  where  $\frac{dy}{du} = 5$ ,  $\frac{du}{dx} = 2x$  and  $\frac{du}{dv} = 3$ .

(C4.4)

*Journal*

- The following question was posed to students: Suppose  $y$  changes twice as fast as  $u$  and  $u$  changes three times fast as  $x$ . How does the rate of change of  $y$  compare with the rate of change of  $x$ ?

Mary responded and said the rate of change was 5, David thought it was 8 and Lynn said it was 6. Ask students who they agree with and why.

(C4.4)

**Resources/Notes**
**Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**2.5: The Chain Rule**

SB: pp. 114 - 122

IG: pp. 116 - 123

PPL: SteEC\_02\_05.ppt

Video: Example 4

## Calculus Reasoning

### Outcomes

Students will be expected to

C4 Continued...

### Achievement Indicator:

C4.5 Continued

### Elaborations—Strategies for Learning and Teaching

Discuss problems with students where the velocity  $v(t)$  is equal to zero, positive or negative. Similar discussions should also center around acceleration  $a(t)$ .

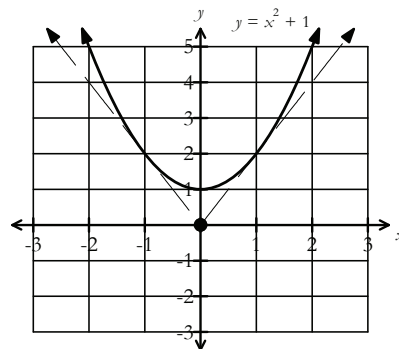
$v(t) = 0$	at rest
$v(t) > 0$	moving right
$v(t) < 0$	moving left
$a(t) = 0$	constant velocity
$a(t) > 0, v(t) > 0$	moving right, speeding up
$a(t) < 0, v(t) > 0$	moving right, slowing down
$a(t) > 0, v(t) < 0$	moving left, slowing down
$a(t) < 0, v(t) < 0$	moving left, speeding up

If the signs of  $v(t)$  and  $a(t)$  are the same (both positive or both negative), then the speed of the object is increasing (i.e.,  $v(t) \cdot a(t) > 0$ ). If the signs of  $v(t)$  and  $a(t)$  are opposite, then the speed of the object is decreasing (i.e.,  $v(t) \cdot a(t) < 0$ ).

When finding the equation of a tangent line (or normal line) to a point on a curve, students can write the equation in either point-slope form or slope-intercept form. They should also solve problems where they have to find the point of tangency. Ask students to solve a problems such as the following:

- Determine the equation of two tangent lines that pass through the origin and are tangent to the parabola  $y = x^2 + 1$ .

Encourage students to sketch a graph to represent this situation so that they can visualize the two tangents.



Students should observe that the point of tangency is unknown. Ask students to label the unknown point of tangency as  $(x, f(x))$  which becomes  $(x, x^2 + 1)$ .

---

**General Outcome: Develop introductory calculus reasoning.**


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**Suggested Assessment Strategies**
*Paper and Pencil*

- Ask students to answer the following:
  - (i) A particle moves left and right along the  $x$ -axis, its position is defined by the position function  $x(t) = t^3 - 12t^2 + 36t - 30$ ,  $t \geq 0$ .
    - (a) Determine the velocity and acceleration functions.
    - (b) When is the velocity zero? When is the acceleration zero?
    - (c) During what time intervals is the velocity positive? Explain what it means.
    - (d) During what time interval is the velocity negative? Explain what it means.
    - (e) When is the particle speeding up? Slowing down?
    - (f) Provide an overall description of what is happening to the particle.

(C4.5)

- (ii) A demographer develops the function  $P(x) = 12500 + 320x - 0.25x^3$  to represent the population of the town Tanville  $x$  years from now.

- (a) Determine the population of the town today.
- (b) Predict the instantaneous rate of change of the population during year 3.
- (c) Determine the average rate of change in the population between years 2 and 7.
- (d) In what year can you expect the population to increase by 245 people?
- (e) Is the rate of change in the population increasing or decreasing? Explain.

(C4.5)

**Resources/Notes**
**Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**2.3: Basic Differentiation  
Formulas**

SB: pp. 95 - 107  
 IG: pp. 102 - 108  
 PPL: SteEC\_02\_03.ppt  
 Video Examples: 9

**2.4: The Product and Quotient  
Rules**

SB: pp. 107 - 114  
 IG: pp. 109 - 115  
 PPL: SteEC\_02\_04.ppt

**2.5: The Chain Rule**

SB: pp. 114 - 122  
 IG: pp. 116 - 123  
 PPL: SteEC\_02\_05.ppt

## Calculus Reasoning

### Outcomes

*Students will be expected to*

**C4 Continued...**

**Achievement Indicator:**

*C4.5 Continued*

### Elaborations – Strategies for Learning and Teaching

Students can now write the slope of the tangent using two points  $(0, 0)$  and  $(x, x^2+1)$ .

$$\frac{y_2 - y_1}{x_2 - x_1} = m_{\text{tangent line}}$$

$$\frac{x^2 + 1 - 0}{x - 0} = f'(x)$$

$$\frac{x^2 + 1}{x} = 2x$$

$$2x^2 = x^2 + 1$$

$$x^2 = 1$$

$$x = \pm 1$$

Students should now be able to determine the points of tangency by substituting the  $x$  values back into the function  $f(x) = x^2 + 1$  (i.e.,  $(1, 2)$  and  $(-1, 2)$ ). There are two strategies they can use to find the slope of the tangent line: the slope formula or the derivative of the curve. Two points for each tangent line are now known. Students can, therefore, determine the slope for each line and write their equations.

Alternatively, the slopes of the tangent lines can be determined using the first derivative  $f'(x) = 2x$ .

$$m_{\text{tangent line on left}} = f'(-1) = -2 \qquad m_{\text{tangent line on right}} = f'(1) = 2$$

Students can write the equation of the tangent lines using the slope and the point  $(0, 0)$ :

$$(y - 0) = -2(x - 0)$$

$$y = -2x$$

$$(y - 0) = 2(x - 0)$$

$$y = 2x$$

Ask students to reflect on the different strategies and choose the method they prefer to use.



---

**General Outcome: Develop introductory calculus reasoning.**


---

**Suggested Assessment Strategies**
*Paper and Pencil*

- Ask students to determine the equation of the tangent line to the curve at the given value.

(i)  $y = x^3 - x^2 + x - 1$ , at  $x = 1$

(ii)  $y = 7\sqrt{x} - 3x$ , at  $x = 1$

(iii)  $y = x + \frac{6}{x}$ , at  $x = 2$

(C4.5)

- Ask students to answer the following:

(i) Find the equation of the normal line to the curve  $s(t) = t \cdot \sqrt[3]{2t + 5}$  at  $t = 11$ .

(ii) Find the equation of both tangent lines to the parabola  $y = x^2 + x$  that passes through  $(2, -3)$ . Students should sketch the curve and the tangents.

(C4.5)

- Ask students to answer the following:

(i) Determine the three points where the tangent lines to the curve  $y = x^4 - 2x^2 + 4$  are horizontal.

(ii) Determine the value of  $k$ , if the tangent to  $y = 2x^3 + kx^2 - 3$  at  $x = 2$  has a slope of 4.

(ii) Determine the coordinates of the point(s) where the tangent to  $y = x^3 + x + 2$  at  $(1, 4)$  intersects the curve again.

(C4.5)

**Resources/Notes**
**Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**2.3: Basic Differentiation  
Formulas**

SB: pp. 95 - 107

IG: pp. 102 - 108

PPL: SteEC\_02\_03.ppt

Video Examples: 9

**2.4: The Product and Quotient  
Rules**

SB: pp. 107 - 114

IG: pp. 109 - 115

PPL: SteEC\_02\_04.ppt

**2.5: The Chain Rule**

SB: pp. 114 - 122

IG: pp. 116 - 123

PPL: SteEC\_02\_05.ppt

## Calculus Reasoning

### Outcomes

*Students will be expected to*

**C5 Determine the derivative of a relation, using implicit differentiation.**

[C, CN, PS, R, V]

### Achievement Indicator:

*C5.1 Determine the derivative of an implicit relation.*

### Elaborations—Strategies for Learning and Teaching

Implicit differentiation is a powerful and important technique. It is used in applications such as related rates and deriving rules for differentiating inverse functions.

It is not always possible for a relation to be written as an explicit function of one variable. When the dependent variable is not explicitly written in terms of the independent variable, then implicit differentiation should be used.

Students have been describing functions by expressing one variable explicitly in terms of another variable (i.e.,  $y = x^2$ ,  $y = \frac{3x-1}{x+2}$ ). Other functions are defined implicitly by a relation between  $x$  and  $y$  such as  $x^2 + y^2 = 100$ . It is possible to solve this equation for  $y$  to get  $y = \pm\sqrt{100 - x^2}$ . If asked to find the slope of the tangent line to the circle at the point  $(6, 8)$ , students could differentiate  $y = \sqrt{100 - x^2}$  and substitute in  $x = 6$ . Introduce students to another method called implicit differentiation. They differentiate both sides of the equation with respect to  $x$  and then solve the resulting equation for  $\frac{dy}{dx}$ . Ask students to differentiate the equation using both methods.

$$y = \sqrt{100 - x^2}$$

$$\frac{dy}{dx} = \frac{1}{2}(100 - x^2)^{-\frac{1}{2}}(-2x)$$

$$\frac{dy}{dx} = -\frac{x}{\sqrt{100 - x^2}}$$

$$x^2 + y^2 = 100$$

$$2x \frac{dx}{dx} + 2y \frac{dy}{dx} = 0$$

$$2y \frac{dy}{dx} = -2x \frac{dx}{dx}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Students should conclude through the use of substitution that  $\frac{dy}{dx} = -\frac{3}{4}$ . They should also notice that the derivative expression for explicit differentiation involves  $x$  only, while the derivative expression for implicit differentiation may involve both  $x$  and  $y$ .

Whether students solve by implicit differentiation or by first solving the given equation for  $y$ , encourage them to choose the most efficient method. There are, however, examples where it is impossible to solve the equation for  $y$  as an explicit function of  $x$ . In these cases, the method of implicit differentiation becomes the only choice for solving for  $y'$ .

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**General Outcome: Develop introductory calculus reasoning.**


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**Suggested Assessment Strategies**
*Paper and Pencil*

- Ask students to find  $\frac{dy}{dx}$  for the following:
  - (i)  $xy = 1$
  - (ii)  $x^3 + 2x + y + y^3 = 3$
  - (iii)  $x^2 + xy + y^2 = 5$
- Ask students to find  $\frac{du}{dx}$  for  $2x^3 - ux - u^2 = ux^2$ .
- Find  $y'$  for the following:
  - (i)  $x^2y^2 - y^3 = xy$
  - (ii)  $\sqrt{x+y} + \sqrt{xy} = 6$

(C5.1)

**Resources/Notes**
**Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**2.6: Implicit Differentiation**

SB: pp. 123- 128

IG: pp. 124 - 130

PPL: SteEC\_02\_06.ppt

Video Examples: 1 and 2

**Calculus Reasoning****Outcomes**

*Students will be expected to*

**C5 Continued...**

**Achievement Indicators:**

*C5.2 Determine the equation of the tangent and normal line to the graph of a relation at a given point.*

*C5.3 Determine the second derivative of a relation, using implicit differentiation.*

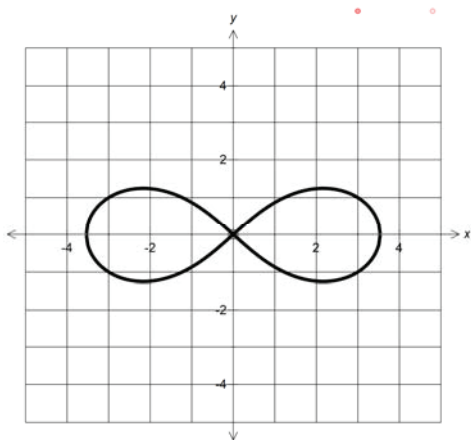
**Elaborations – Strategies for Learning and Teaching**

Using implicit differentiation, students should determine the equation of the tangent and normal line to the graph of a relation at a given point. Ask students to determine, for example, the equation of a tangent and normal line to the curve  $3x^2 - 2xy + xy^3 = 7$  at  $(1, 2)$ .

Students should solve problems where they calculate the second derivative of an implicitly defined relationship. Given the equation  $x^2 - xy - y^2 - 2y = 0$ , for example, ask students to obtain  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$  at the point  $(4, 2)$  on the curve defined by the equation. Students should be aware that the first derivative represents the slope of the tangent line at a point and the second derivative can be interpreted as the rate of change of the slope of the tangent line.

**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Paper and Pencil*

- The curve  $2(x^2 + y^2)^2 = 25(x^2 - y^2)$  shown below is called a lemniscate. Ask students to determine the equation of the tangent line to this curve at  $(3,1)$ .



(C5.2)

- Ask students to determine the equation of the tangent to the curve  $3x^5 + x^4y + y^5 = 49$  at the point  $(1,2)$ .

(C5.2)

- Ask students to show that  $y'' = -\frac{20x}{y^5}$  where  $x^3 + y^3 = 10$ .

(C5.3)

**Resources/Notes****Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**2.6: Implicit Differentiation**

SB: pp. 123- 128

IG: pp. 124 - 130

PPL: SteEC\_02\_06.ppt

Video Examples: 1 and 2



Applications of Derivatives

Suggested Time: 20 Hours

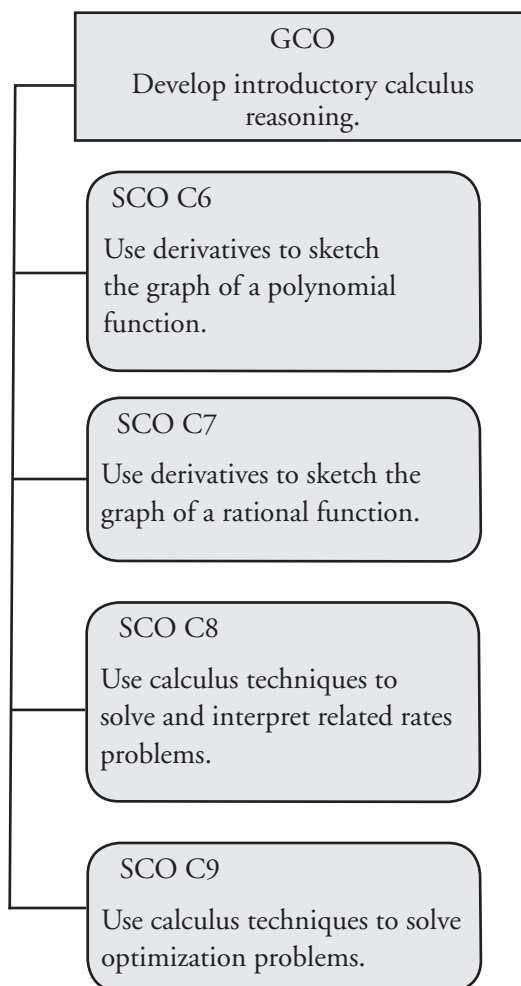
# Unit Overview

## Focus and Context

Using the first derivative test, students will determine the extreme values of a function and the intervals where the function increases and decreases. They will determine the concavity of the graph and how to find the points of inflection using the second derivative test. Characteristics such as domain, intercepts and asymptotes should also be discussed. Using all of this information, students will sketch the graph of polynomial and rational functions.

Students will use calculus techniques to solve and interpret related rates problems and optimization problems.

## Outcomes Framework





## SCO Continuum

Mathematics 2200	Mathematics 3200	Calculus 3208
Calculus		
RF4 Analyze quadratic functions of the form $y = ax^2 + bx + c$ , $a \neq 0$ , to identify characteristics of the corresponding graph, including: <ul style="list-style-type: none"> <li>• vertex</li> <li>• domain and range</li> <li>• direction of opening</li> <li>• axis of symmetry</li> <li>• <math>x</math>- and <math>y</math>-intercepts</li> </ul> and to solve problems. [CN, PS, R, T, V]	RF11 Graph and analyze polynomial functions (limited to polynomial functions of degree $\leq 5$ ).  [C, CN, T, V]	C6 Use derivatives to sketch the graph of a polynomial function. [C, CN, PS, R, T, V]  C7 Use derivatives to sketch the graph of a rational function. [C, CN, PS, R, T, V]  C8 Use calculus techniques to solve and interpret related rates problems. [C, CN, ME, PS, R]  C9 Use calculus techniques to solve optimization problems. [C, CN, ME, PS, R]

## Mathematical Processes

[C] Communication	[PS] Problem Solving
[CN] Connections	[R] Reasoning
[ME] Mental Mathematics and Estimation	[T] Technology
	[V] Visualization

## Calculus Reasoning

### Outcomes

Students will be expected to

**C6 Use derivatives to sketch the graph of a polynomial function.**

[C, CN, PS, R, T, V]

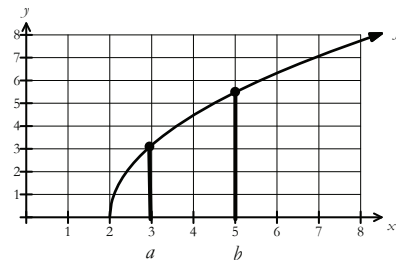
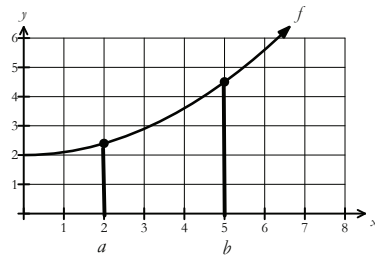
### Achievement Indicator:

C6.1 Use  $f'(x)$  to identify the critical numbers, relative and absolute extrema, and intervals of increase and decrease.

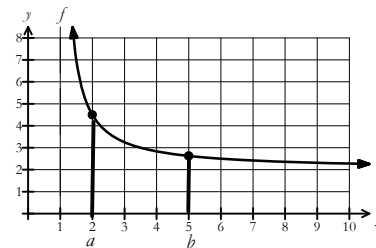
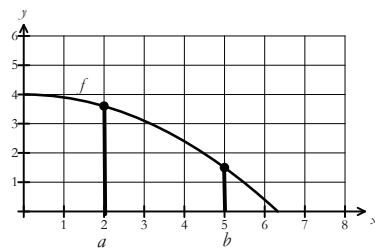
### Elaborations – Strategies for Learning and Teaching

Students will describe the key features of a polynomial function given information about its first and second derivative. Using the first derivative, they will identify the intervals of increase and decrease, critical numbers, and relative and absolute extrema. Using the second derivative, they will identify the hypercritical numbers, intervals of concavity and points of inflection. Students will sketch the graph of the function using information obtained from the function and its derivatives.

Students should be able to look at a graph and identify where that function is increasing and decreasing. This would be a good opportunity to introduce them to the formal definition of an increasing and decreasing function using graphs such as the ones shown below.



Moving left to right along the graph, a function,  $f$ , is increasing if the  $y$ -coordinate increases in value (ie. if  $f(b) > f(a)$  whenever  $b > a$ ).



Moving left to right along the graph, a function,  $f$ , is decreasing if the  $y$ -coordinate decreases in value (ie. if  $f(b) < f(a)$  whenever  $b > a$ ).

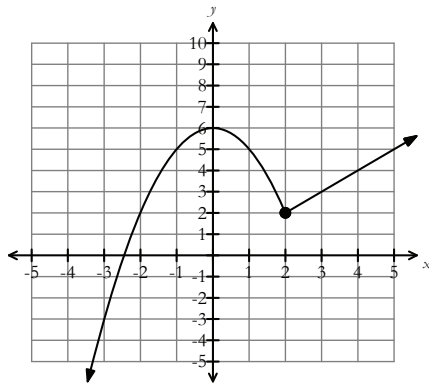
**General Outcome: Develop introductory calculus reasoning.**

**Suggested Assessment Strategies**

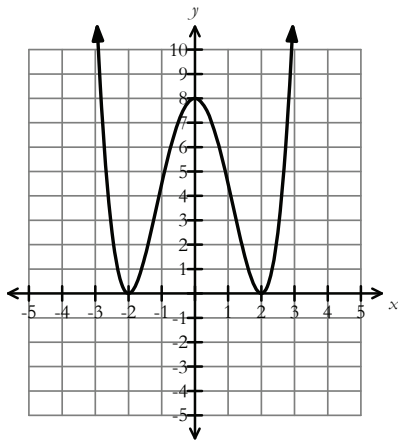
*Paper and Pencil*

- Ask students to determine the intervals of increase and decrease for the following:

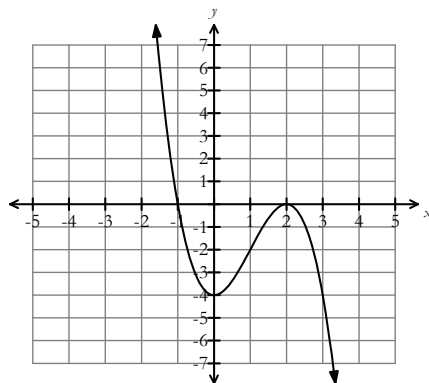
(i)



(ii)



(iii)



**Resources/Notes**

**Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**3.1: Maximum and Minimum Values**

Student Book (SB): pp. 145 - 152  
Instructor's Guide (IG): pp. 151 - 156

Power Point Lecture (PPL):  
SteEC\_03\_01.ppt

Video Examples: 4, 5 and 6

**3.3: Derivatives and the Shapes of Graphs**

SB: pp. 158 - 166

IG: pp. 163 - 174

PPL: SteEC\_03\_03.ppt

Video Examples: 1

(C6.1)

**Calculus Reasoning**

**Outcomes**

*Students will be expected to*

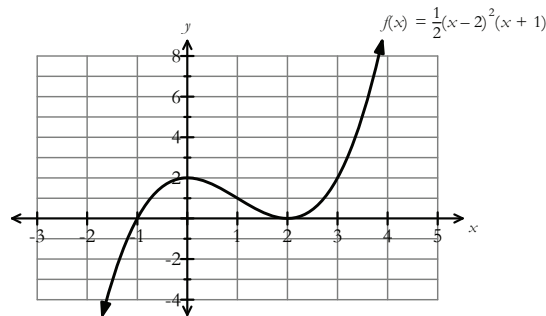
**C6 Continued...**

**Achievement Indicator:**

C6.1 *Continued*

**Elaborations – Strategies for Learning and Teaching**

Students are aware that the slope of the tangent line at a point is given by the value of the derivative at that point. Provide students with the graph of a function such as  $f(x) = \frac{1}{2}(x - 2)^2(x + 1)$  and use the following questions to promote student discussion:



- Where is  $f(x)$  increasing?
- What do you notice about the slopes of the tangents when  $f(x)$  is increasing?
- Where is  $f(x)$  decreasing?
- What do you notice about the slopes of the tangents when  $f(x)$  is decreasing?
- Where is the slope of the tangent equal to zero?

Students should observe that on intervals where the tangent lines have positive slope, the function is increasing while on intervals where the tangent lines have negative slope the function is decreasing. The tangent lines have a zero slope at  $x = 0$  and  $x = 2$ , which is where the graph changes direction.

Provide students the generalization for any function  $f(x)$  which is differentiable on a given interval:

- If  $f'(x) > 0$  for all  $x$  in an interval, then  $f(x)$  is increasing on that interval.
- If  $f'(x) < 0$  for all  $x$  in an interval, then  $f(x)$  is decreasing on that interval.
- If  $f'(c) = 0$ , then the graph of  $f(x)$  has a horizontal tangent at  $x = c$ .

**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Journal*

- Ask students to respond to the following:
  - (i) How can you estimate, from the graph of a polynomial function, the intervals where the function is increasing? Decreasing?
  - (ii) What is always true about the slopes of all tangent lines on a section of a curve that is rising? Falling?
  - (iii) Graph a function that is increasing on the interval  $-2 < x < 2$ , decreasing on the interval  $2 < x < 4$ , and increasing on the interval  $4 < x < 7$ . Draw a smooth curve. Suggest the degree of a polynomial function that fits this description.

(C6.1)

**Resources/Notes****Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**3.1: Maximum and Minimum Values**

SB: pp. 145 - 152

IG: pp. 151 - 156

PPL: SteEC\_03\_01.ppt

Video Examples: 4, 5 and 6

**3.3: Derivatives and the Shapes of Graphs**

SB: pp. 158 - 166

IG: pp. 163 - 174

PPL: SteEC\_03\_03.ppt

Video Examples: 1

## Calculus Reasoning

### Outcomes

Students will be expected to

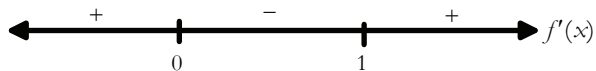
C6 Continued...

**Achievement Indicator:**

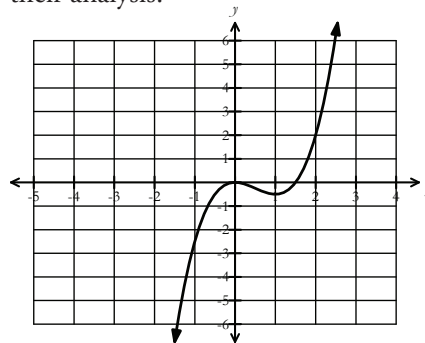
C6.1 Continued

### Elaborations – Strategies for Learning and Teaching

When provided with the graph of a function, students should be able to observe where the function is increasing and decreasing. They should now be asked to determine where the function is increasing or decreasing given only the equation of the function. Ask students to algebraically determine the intervals of increase and decrease for polynomial functions such as  $f(x) = x^3 - \frac{3}{2}x^2$ . Solving the equation  $f'(x) = 0$ , they are able to determine value(s) of  $x$  where possible turning points may exist. A sign diagram can then be completed to determine where the derivative (i.e.,  $f'(x) = 3x^2 - 3x$ ) is positive or negative and thus determine the intervals of increase and decrease.



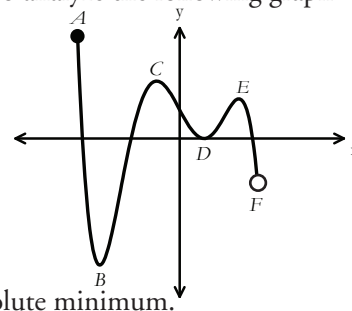
Students should notice that the function increases on the interval  $(-\infty, 0)$  and  $(1, \infty)$  and decreases on the interval  $(0, 1)$ . A graph confirms their analysis.



This would be a good opportunity to ask students why the  $x$ -values, 0 and 1, are not included in the interval.

It is important for students to understand the difference between absolute (global) extreme values and relative (local) extreme values. Inform students that the local maximum or minimum are the highest or lowest point on the graph within a certain interval. The absolute maximum and minimum values are found at local extrema or at the endpoints on the interval. Ask students to analyze the following graph with a focus on the following:

- Point A is an absolute maximum.
- Point B is both a local and absolute minimum.
- Point C and Point E are local maximum values.
- Point D is a local minimum.
- Point F is neither a local nor an absolute minimum.



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**General Outcome: Develop introductory calculus reasoning.**


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**Suggested Assessment Strategies**
*Paper and Pencil*

- Ask students to algebraically determine the intervals of increase and decrease for the following functions.

(i)  $f(x) = 2x^3 - 3x^2$

(ii)  $f(x) = 3x^4 - 6x^2$

(iii)  $f(x) = x^3 - 3x^2 - 9x + 5$ .

(C6.1)

**Resources/Notes**
**Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**3.1: Maximum and Minimum Values**

SB: pp. 145 - 152  
 IG: pp. 151 - 156  
 PPL: SteEC\_03\_01.ppt  
 Video Examples: 4, 5 and 6

**3.3: Derivatives and the Shapes of Graphs**

SB: pp. 158 - 166  
 IG: pp. 163 - 174  
 PPL: SteEC\_03\_03.ppt  
 Video Examples: 1

## Calculus Reasoning

### Outcomes

Students will be expected to

C6 Continued...

#### Achievement Indicator:

C6.1 Continued

### Elaborations—Strategies for Learning and Teaching

Critical numbers of a function  $f(x)$  are defined as any values of  $x$  for which

- $f'(x) = 0$ , or
- $f'(x)$  is undefined.

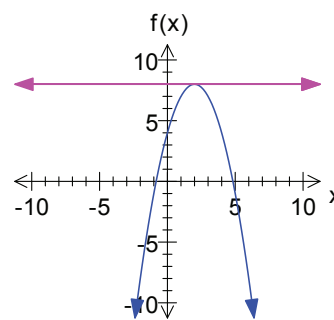
There are several cases that students should consider in terms of function behaviour at critical numbers. When  $f'(c) = 0$ , there is a horizontal tangent, which indicates one of the following cases:

- (i) a local maximum

$$f(x) = -x^2 + 4x + 4$$

$$f'(x) = -2x + 4$$

$$f'(2) = 0$$

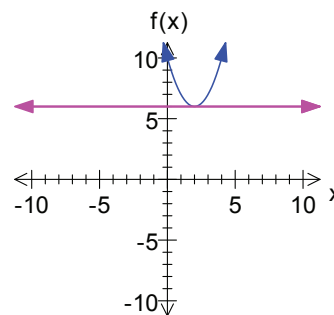


- (ii) a local minimum

$$f(x) = x^2 - 4x + 10$$

$$f'(x) = 2x - 4$$

$$f'(2) = 0$$

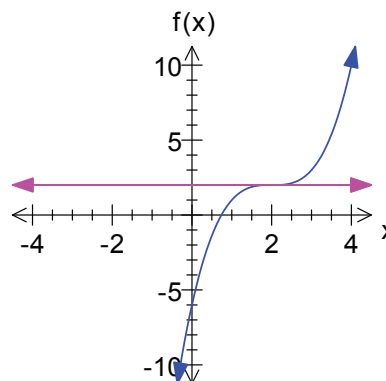


- (iii) neither, since the sign of  $f'(x)$  does not change through  $(2,2)$

$$f(x) = x^3 - 6x^2 + 12x - 6$$

$$f'(x) = 3x^2 - 12x + 12$$

$$f'(2) = 0$$



When  $f'(x)$  is undefined at  $x = a$ , the function is not differentiable. Students have previously learned that  $f'(x)$  is undefined at cusps, corners, discontinuities, or where  $f(x)$  has vertical tangents.

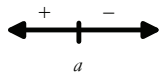
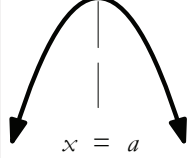
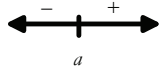
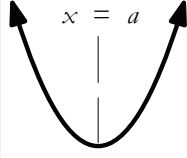
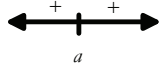
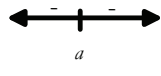
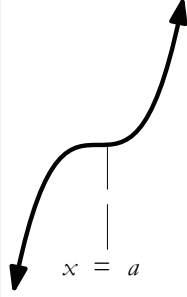
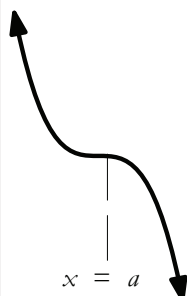


**General Outcome: Develop introductory calculus reasoning.**

**Suggested Assessment Strategies**

*Paper and Pencil*

- Ask students to write a summary of critical points for which  $f'(a) = 0$ . A sample is shown below:

sign diagram of $f'(x)$ near $x = a$	description	shape of curve near $x = a$
	If $f'(x)$ changes from positive to negative at $a$ and $f(a)$ exists, then $f(a)$ is a local maximum.	
	If $f'(x)$ changes from negative to positive at $a$ and $f(a)$ exists, then $f(a)$ is a local minimum.	
 	If $f'(x)$ does not change sign, then there is neither a local maximum nor local minimum at $a$ .	 

(C6.1)

**Resources/Notes**

**Authorized Resource**

*Single Variable Essential Calculus Second Edition*

**3.1: Maximum and Minimum Values**

SB: pp. 145 - 152  
 IG: pp. 151 - 156  
 PPL: SteEC\_03\_01.ppt  
 Video Examples: 4, 5 and 6

**3.3: Derivatives and the Shapes of Graphs**

SB: pp. 158 - 166  
 IG: pp. 163 - 174  
 PPL: SteEC\_03\_03.ppt  
 Video Examples: 1

## Calculus Reasoning

### Outcomes

Students will be expected to

C6 Continued...

#### Achievement Indicator:

C6.1 Continued

### Elaborations – Strategies for Learning and Teaching

Critical numbers are used to find the absolute maximum and minimum on a given interval. As students determine the absolute maximum and minimum values of a function similar to  $f(x) = -x^3 + 27x + 5$  on the interval  $[0, 4]$ , they should use the following guidelines:

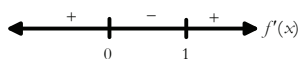
- Determine the critical numbers in the interval and compute functional values at these points.
- Determine the functional values at the endpoints  $f(0)$  and  $f(4)$ .
- The largest value is the absolute maximum; the smallest value is the absolute minimum.

Using the Power Rule, students determine  $f'(x) = -3x^2 + 27$ . Setting  $f'(x) = 0$ , they should obtain  $x = \pm 3$ . Students only need to consider  $x = 3$  since the value  $x = -3$  is outside the interval  $[0, 4]$ . To find the absolute maximum and minimum, they should evaluate  $f(x)$  at the critical numbers and the value of  $f(x)$  at the endpoints of the interval. In particular, they need to compute  $f(x)$  for  $x = 0, 3$  and  $4$ . Since  $f(0) = 5, f(3) = 59$  and  $f(4) = 49$ , students should conclude the absolute maximum is at  $x = 3$  (i.e., the critical number) and the absolute minimum is at  $x = 0$  (i.e., the endpoint).

Students are aware that local maximum or minimum values can occur at a critical number. A sign diagram is a useful tool when trying to determine the local extrema. Applying the first derivative test, students can determine the critical point(s) and check where  $f'(x)$  changes sign. Ask students to revisit the example  $f(x) = x^3 - \frac{3}{2}x^2$  that was discussed earlier to determine the local maximum or minimum algebraically.

$$f'(x) = 3x^2 - 3x$$

critical values:  $x = 0$  and  $x = 1$

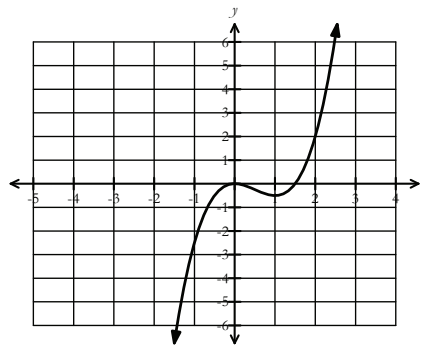


increasing:  $(-\infty, 0)$  and  $(1, \infty)$

decreasing:  $(0, 1)$

local maximum  $(0, 0)$

local minimum  $(1, -\frac{1}{2})$



To find the  $y$ -value of the local minimum/maximum value, students sometimes mistakenly substitute the critical number(s),  $c$ , back into  $f'(x)$  rather than  $f(x)$ .

**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Paper and Pencil*

- Ask students to find the absolute extrema for  $y = f(x)$  on the indicated intervals:
  - (i)  $f(x) = 2x^3 - 3x^2 - 36x + 62$  on  $[-3, 4]$
  - (ii)  $f(x) = (x^2 - 1)^{\frac{1}{3}}$  on  $[-2, 2]$

(C6.1)
  
- Ask students to algebraically determine the local extrema, if they exist, for the following functions:
  - (i)  $f(x) = x^3 - 3x + 1$
  - (ii)  $f(x) = x^4 - 8x^2 - 10$

They should verify their solution using graphing technology.

(C6.1)
  
- Ask students to answer the following for the function  $f(x) = 2x^3 - 3x^2 - 12x + 5$ .
  - (i) Determine the critical numbers of  $f(x)$ .
  - (ii) Determine any local extrema of  $f(x)$ .
  - (iii) Determine the absolute extrema of  $f(x)$  in the interval  $[-2, 4]$ .

(C6.1)

**Resources/Notes****Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**3.1: Maximum and Minimum Values**

SB: pp. 145 - 152  
 IG: pp. 151 - 156  
 PPL: SteEC\_03\_01.ppt  
 Video Examples: 4, 5 and 6

**3.3: Derivatives and the Shapes of Graphs**

SB: pp. 158 - 166  
 IG: pp. 163 - 174  
 PPL: SteEC\_03\_03.ppt

**Calculus Reasoning**

**Outcomes**

*Students will be expected to*

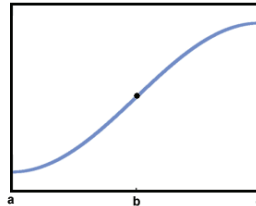
**C6 Continued...**

**Achievement Indicator:**

C6.2 Use  $f''(x)$  to identify the hypercritical numbers, points of inflection and intervals of concavity.

**Elaborations – Strategies for Learning and Teaching**

To introduce students to the concept of concavity, provide students with a graph such as the following:

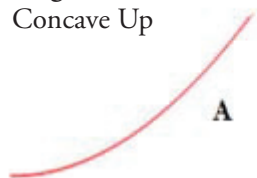


Students should notice that  $f(x)$  is increasing on  $[a, c]$ . However, they should be questioning the behaviour prior to point  $b$  and how it is different from its behaviour after point  $b$ . A section of the graph of  $f(x)$  is considered to be concave up if its slope (i.e.,  $f'(x)$ ) increases as  $x$  increases. A section of the graph of  $f(x)$  is considered to be concave down if its slope decreases as  $x$  increases.

To further aid in the understanding of the concept of concavity, students could analyze the following graphs. They should think about where the tangent lines are in relation to each curve and how the slope of the tangent lines are changing for each curve.

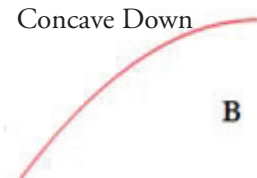
• **Increasing Function**

Concave Up



**A**

Concave Down



**B**

Given that a curve is increasing, students need further information to determine which of the two shapes shown they should draw. Suppose, for example, the graphs represent the balance in a bank account. Both indicate that the balance is growing. Ask students which graph they would want to describe their account balance and why. They should make reference to the rate of growth in both graphs. Point out to students that although all of the tangent lines have positive slope (i.e., since  $f'(x) > 0$ ), the slopes of the tangent lines are increasing in Figure A, while those in Figure B are decreasing.

Students should be able to connect concavity with what they already know about increasing and decreasing functions:

- if the slope of the tangent line increases, this means that the derivative  $f'$  is an increasing function and therefore its derivative  $f''$  is positive.
- if the slope of the tangent line decreases, this means that the derivative  $f'$  decreases and therefore its derivative  $f''$  is negative.

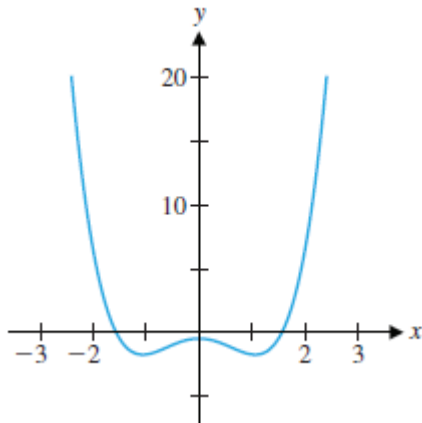
**General Outcome: Develop introductory calculus reasoning.**

**Suggested Assessment Strategies**

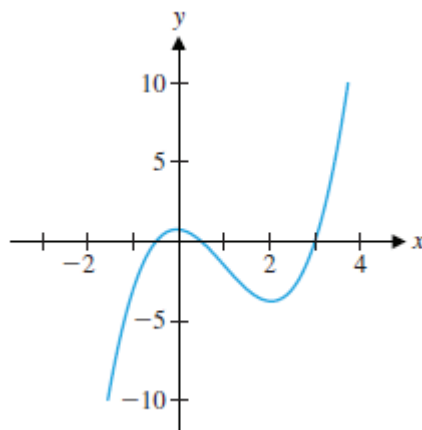
*Paper and Pencil*

- Ask students to estimate the intervals where the function is concave up or down.

(i)



(ii)



(C6.2)

*Interview*

- Ask students to respond to the following:  
Angela said the following, "For concave up curves, the function must be increasing". Is she correct and explain your reasoning.

(C6.2)

**Resources/Notes**

**Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**3.3: Derivatives and the Shapes of Graphs**

SB: pp. 158 - 166

IG: pp. 163 - 174

PPL: SteEC\_03\_03.ppt

**Suggested Resource**

Resource Link:

[www.k12pl.nl.ca/curr/10-12/math/3208/res-link/appsderivs.html](http://www.k12pl.nl.ca/curr/10-12/math/3208/res-link/appsderivs.html)

- concavity and increasing or decreasing functions

## Calculus Reasoning

### Outcomes

Students will be expected to

C6 Continued...

**Achievement Indicator:**

C6.2 Continued

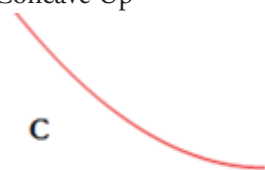
### Elaborations – Strategies for Learning and Teaching

Students should then analyze decreasing functions.

- **Decreasing Function**

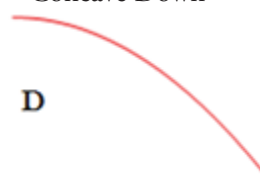
Concave Up

**C**



Concave Down

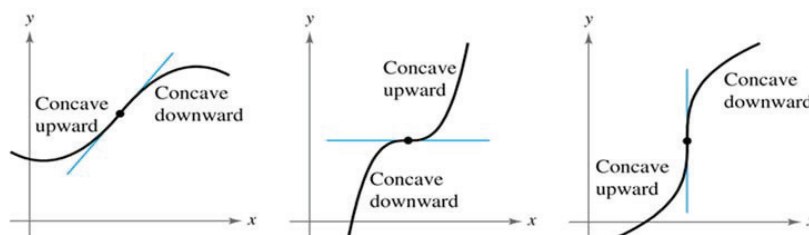
**D**



Although all of the tangent lines have negative slope (i.e., since  $f'(x) < 0$ ), the slopes of the tangent lines are increasing in Figure C, while those in Figure D are decreasing.

Reinforce to students that if a function is concave up, the curve lies above its tangent lines. If a function is concave down, the curve lies below its tangent lines.

Provide the following graphs to students to illustrate that the concavity of a function changes at a point of inflection.



It is important for students to notice that inflection points occur where the direction of concavity changes (i.e., concave upward to concave downward, or vice versa).

To determine the intervals where a function is concave upward or concave downward, students should first find the domain values where  $f''(x) = 0$  or  $f''(x)$  does not exist. These are referred to as the hypercritical values. Students should test all intervals around these values in the second derivative of the function. A function is said to be concave upward on an interval if  $f''(x) > 0$  within the interval and concave downward on an interval if  $f''(x) < 0$  within the interval. If  $f''(x)$  changes sign, the hypercritical value is a point of inflection. It marks the transition between concave up and concave down. As with the first derivative test for local extrema, there is no guarantee that the second derivative will change signs, and therefore, it is essential to test each interval around the values for which  $f''(x) = 0$  or does not exist.

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**General Outcome: Develop introductory calculus reasoning.**


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**Suggested Assessment Strategies**
*Paper and Pencil*

- Ask students to determine the intervals of concavity and determine any inflection points.

(i)  $f(x) = 6x^2 - 12x + 1$

(ii)  $f(x) = x^4 + x^3 - 3x^2 + 1$

(C6.2)

- Ask students to show that the function  $f(x) = x^4$  satisfies  $f''(0) = 0$  but has no inflection point.

(C6.2)

**Resources/Notes**
**Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**3.3: Derivatives and the Shapes of Graphs**

SB: pp. 158 - 166

IG: pp. 163 - 174

PPL: SteEC\_03\_03.ppt

Video Examples: 4 and 5

### Calculus Reasoning

#### Outcomes

Students will be expected to

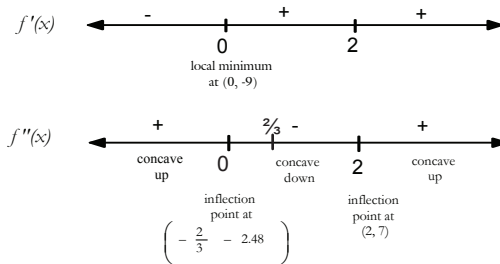
C6 Continued...

Achievement Indicator:

C6.2 Continued

#### Elaborations – Strategies for Learning and Teaching

Ask students to determine where the graph of a function such as  $f(x) = 3x^4 - 16x^3 + 24x^2 - 9$ , is concave up and concave down, determine any inflection points and draw a graph to support their findings. Students should first calculate the first and second derivative. Setting  $f'(x) = 0$  results in  $x = 0$  and  $x = 2$ . Setting  $f''(x) = 0$  results in  $x = 2$  and  $x = \frac{2}{3}$ .



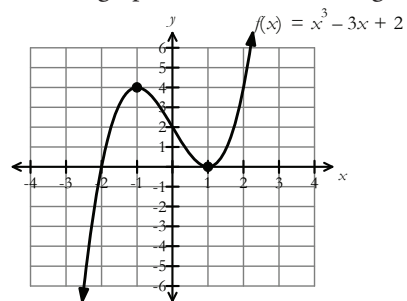
Use the sign diagram to identify the inflection points and intervals of concavity. Students should then graph the function.

Ask students, "If a point of inflection must occur at a hypercritical number, are all hypercritical numbers classified as points of inflection?"

The second derivative provides students another way to classify critical points as local maxima or local minima. Go through the following points with students:

- If  $f'(c) = 0$  and  $f''(c) > 0$  and  $f$  is concave upward, then  $f$  has a local minimum at  $c$ .
- If  $f'(c) = 0$  and  $f''(c) < 0$  and  $f$  is concave downward, then  $f$  has a local maximum at  $c$ .
- If  $f''(c) = 0$  then  $x = c$  can be a local maximum, local minimum or neither (i.e., students must use the first derivative test to classify the critical point).

Ask students to determine the extrema using the second derivative test for a function such as  $f(x) = x^3 - 3x + 2$ . Since  $f'(x) = 3x^2 - 3$ , the critical numbers are  $x = \pm 1$ . The value of  $f''(1) = 6$ . This implies there is a local minimum at  $(1, 0)$ . The value of  $f''(-1) = -6$ . This implies there is a local maximum at  $(-1, 4)$ . Ask students to graph the function, using technology, to verify their results.





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**General Outcome: Develop introductory calculus reasoning.**


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**Suggested Assessment Strategies**
*Paper and Pencil*

- Ask students to sketch the following three functions:  $y = x^4$ ,  $y = -x^4$  and  $y = x^3$ . They should verify that all three functions have a critical point at  $x = 0$  and that the second derivative of all functions is zero at  $x = 0$ . Ask students to determine which function has a relative minimum, relative maximum or neither.

(C6.2)

**Resources/Notes**
**Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**3.3: Derivatives and the Shapes of Graphs**

SB: pp. 158 - 166  
 IG: pp. 163 - 174  
 PPL: SteEC\_03\_03.ppt  
 Video Examples: 4 and 5

**3.4: Curve Sketching**

SB: pp. 166 - 173  
 IG: pp. 175 - 184  
 PPL: SteEC\_03\_04.ppt  
 Video: Example 1  
 Tools for Enriching Calculus  
 Animations (TEC): Module 3.4:  
 Using Derivatives to Sketch a  
 Graph of  $f$

**Calculus Reasoning****Outcomes**

*Students will be expected to*

**C6 Continued...**

**Achievement Indicators:**

C6.3 *Sketch the graph of  $f(x)$  using information obtained from the function and its derivatives.*

C6.4 *Use the given function  $f(x)$  to determine its features such as intercepts and the domain.*

**Elaborations – Strategies for Learning and Teaching**

In Mathematics 3200, students graphed polynomial functions using the degree of the function, the leading coefficient, the  $x$ -intercepts, and the  $y$ -intercept (RF11). Students will continue to use the equation of the function to determine these characteristics. Information from the first and second derivative can now be combined to help students refine the graph and make it more accurate. They will now use the intervals of increase or decrease, critical numbers, relative and absolute extrema, hypercritical numbers, intervals of concavity and points of inflection.

**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Paper and Pencil*

- Ask students to answer the following questions for the function  
 $y = x^3 + 3x^2 - 9x - 11$ 
  - State the domain and  $y$ -intercept.
  - Algebraically find the  $x$ -intercepts.
  - Find  $y'$ .
  - Find the critical numbers.
  - State the intervals of increase and decrease.
  - Find the local extrema.
  - Find  $y''$ .
  - Find the hypercritical numbers.
  - State the intervals of concavity. State the points of inflection, if any exist.
  - Sketch the graph.

(C6.3, C6.4)

*Performance*

- Ask students to create a graphic organizer to help them sketch the curve of a polynomial function highlighting the following characteristics:
  - domain of the function
  - intercepts of the function
  - intervals of increase or decrease
  - local maximum and minimum values
  - concavity and points of inflection
  - sketch the curve

(C6.3, C6.4)

- Ask students to sketch the graph of a continuous function that satisfies all of the following conditions:

$$f(0) = f(3) = 0, f(-1) = f(1) = -2$$

$$f'(-1) = f'(1) = 0$$

$$f'(x) < 0 \text{ for } x < -1 \text{ and for } 0 < x < 1$$

$$f'(x) > 0 \text{ for } -1 < x < 0 \text{ and for } x > 1$$

$$\lim_{x \rightarrow \infty} f(x) = 1, \lim_{x \rightarrow -\infty} f(x) = 0$$

Ask students to discuss the number of inflection points on the graph.

(C6.3, C6.4)

**Resources/Notes****Authorized Resource**

*Single Variable Essential Calculus*  
 Second Edition

**3.3: Derivatives and the Shapes of Graphs**

SB: pp. 158 - 166

IG: pp. 163 - 174

PPL: SteEC\_03\_03.ppt

Video Examples: 4 and 5

**3.4: Curve Sketching**

SB: pp. 166 - 173

IG: pp. 175 - 184

PPL: SteEC\_03\_04.ppt

Video: Example 1

TEC: Module 3.4: Using

Derivatives to Sketch a Graph of  $f$ **Suggested Resource**

Resource Link:

[www.k12pl.nl.ca/curr/10-12/math/3208/res-link/appsderivs.html](http://www.k12pl.nl.ca/curr/10-12/math/3208/res-link/appsderivs.html)

- classroom clip of students performing a curve sketching task

## Calculus Reasoning

### Outcomes

Students will be expected to

**C7 Use derivatives to sketch the graph of a rational function.**

[C, CN, PS, R, T, V]

### Achievement Indicators:

C7.1 Use  $f'(x)$  to identify the critical numbers, relative and absolute extrema and intervals of increase and decrease.

C7.2 Use  $f''(x)$  to identify the hypercritical numbers, points of inflection and intervals of concavity.

C7.3 Sketch the graph of  $f(x)$  using information obtained from the function and its derivatives.

C7.4 Use the given function  $f(x)$  to determine its features such as intercepts, asymptotes, points of discontinuity and the domain.

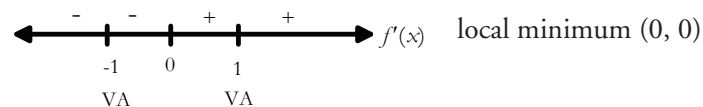
### Elaborations – Strategies for Learning and Teaching

Students will describe the key features of a rational function given information about its first and second derivative. Using the first derivative, they will identify the intervals of increase and decrease, critical numbers, and relative and absolute extrema. Using the second derivative, they will identify the hypercritical numbers, intervals of concavity and points of inflection. Students will sketch the graph of the function using information obtained from the function and its derivatives.

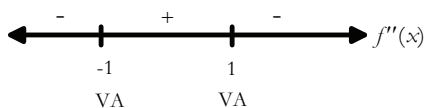
Students should work with rational functions such as  $f(x) = \frac{x^2}{1-x^2}$ . They should notice the function is undefined at  $x = -1$  and  $x = 1$  which are the vertical asymptotes. The  $x$ -intercept is at  $x = 0$  since this is where the numerator equals zero. Students should determine the horizontal asymptote by computing the  $\lim_{x \rightarrow \pm\infty} \frac{x^2}{1-x^2}$ .

As with polynomials, students should apply the first derivative test to determine the critical numbers, determine where the function is increasing and decreasing, and identify any local maxima or minima values.

After applying the Quotient Rule to determine  $f'(x) = \frac{2x}{(1-x^2)^2}$  remind students that  $f'(x) = 0$  when the numerator equals zero. Students should conclude the critical number is  $x = 0$ . In this case, the sign of  $f'(x)$  is determined by the numerator, the product  $2x$ , since the denominator  $(1-x^2)^2$  is always positive. Ask students to determine whether  $f'(x) > 0$  or  $f'(x) < 0$  on each of the intervals between the critical points and the vertical asymptotes.



Students should use the second derivative test to determine the hypercritical numbers, concavity and identify the points of inflection. Ask students to determine the second derivative,  $f''(x) = \frac{2+6x^2}{(1-x^2)^3}$ . They should notice  $f''(x) \neq 0$  for any real value of  $x$ . Students should consider the vertical asymptotes when finding the intervals of concavity.



There is no inflection point since  $x = -1$  and  $x = 1$  are not in the domain. Students should notice the function is concave upward on  $(-1, 1)$  and concave downward on  $(-\infty, -1)$  and  $(1, \infty)$ .

**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Paper and Pencil*

- Ask students to sketch the graph of a function that satisfies all of the following conditions:
  - $f(2) = 3, f(5) = 6, f(4) = 4$
  - $f'(2) = f'(5) = 0$
  - 4 is the only hypercritical number
  - $f'(x) < 0$  for  $x < 2, x > 5$

(C6.3, C6.4)
  
- Ask students to identify the intervals of increase and decrease, using a sign diagram, for  $f(x) = \frac{2x-3}{x^2-6x+9}$ .
 

(C7.1)
  
- Ask students to determine the intervals of concavity and find any inflection points for the function  $f(x) = \frac{6}{x^2+3}$ .
 

(C7.2)
  
- Ask students to answer the following questions for the function  $y = \frac{x^2}{x^2+1}$ .
  - (a) State the domain and  $y$ -intercept.
  - (b) Algebraically find the  $x$ -intercepts.
  - (c) Find  $y'$ .
  - (d) Find the critical numbers.
  - (e) State the intervals of increase and decrease.
  - (f) Find the local extrema.
  - (g) Find  $y''$ .
  - (h) Find the hypercritical numbers.
  - (i) State the intervals of concavity. State the points of inflection, if any exist.
  - (j) Sketch the graph.

(C7.1, C7.2, C7.3, C7.4)

**Resources/Notes****Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**3.1: Maximum and Minimum Values**

SB: pp. 145 - 152  
 IG: pp. 151 - 156  
 PPL: SteEC\_03\_01.ppt

**3.3: Derivatives and the Shapes of Graphs**

SB: pp. 158 - 166  
 IG: pp. 163 - 174  
 PPL: SteEC\_03\_03.ppt

## Calculus Reasoning

### Outcomes

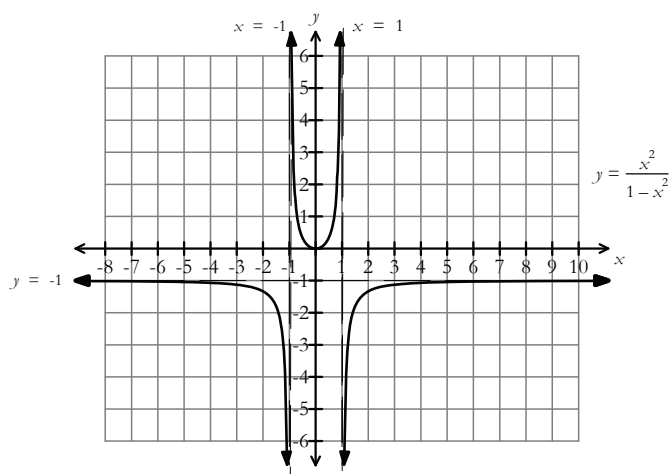
Students will be expected to  
C7 Continued...

### Achievement Indicators:

C7.1, C7.2, C7.3, C7.4  
Continued

### Elaborations – Strategies for Learning and Teaching

Ask students to sketch the graph of the function using the information from the first and second derivative. They should also use the given function to determine features such as intercepts and asymptotes and points of discontinuity. Ask students to confirm their results by sketching the graph of the function using graphing technology.



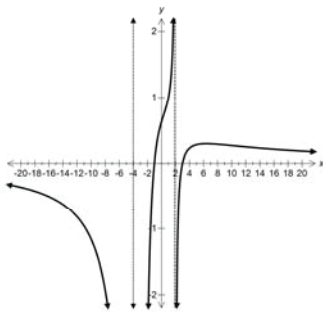
**General Outcome: Develop introductory calculus reasoning.**

**Suggested Assessment Strategies**

*Performance*

- Post four graphs around the room and provide each student with a card containing specific information that matches with one of the graphs. Students try to match their information to one of the posted graphs. Ask them to stand next to the graph. They should notice that other students will join them but have different information. Encourage them to discuss the information on each other's card to validate their work.

Sample Graph:



Sample Cards:

$\lim_{x \rightarrow -4} f(x) = -\infty$	$f'(6) = 0$
	$f''(6) < 0$

(C7.2)

**Resources/Notes**

**Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**3.1: Maximum and Minimum Values**

SB: pp. 145 - 152

IG: pp. 151 - 156

PPL: SteEC\_03\_01.ppt

**3.3: Derivatives and the Shapes of Graphs**

SB: pp. 158 - 166

IG: pp. 163 - 174

PPL: SteEC\_03\_03.ppt

## Calculus Reasoning

### Outcomes

*Students will be expected to*

**C8 Use calculus techniques to solve and interpret related rates problems.**

[C, CN, ME, PS, R]

### Achievement Indicators:

C8.1 *Solve a problem involving related rates drawn from a variety of applications.*

C8.2 *Interpret the solution to a related rates problem.*

### Elaborations – Strategies for Learning and Teaching

Students will solve problems in calculus that require them to find the rate of change of two or more variables that are related to a common variable, namely time. To solve these types of problems, the appropriate rate of change is determined by implicit differentiation with respect to time.

As a preliminary discussion to related rates, the following examples could be used to illustrate how one thing changes with respect to another thing changing.

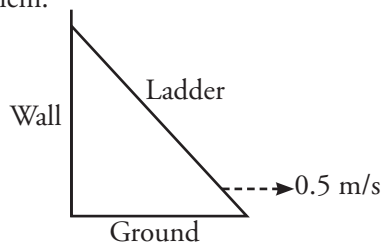
- When inflating a balloon, the volume changes with respect to the radius or the surface area changes with respect to the radius.
- In physics, velocity is the change in displacement with respect to time and acceleration is the change in velocity with respect to time.
- In biology, there is a rate of change of population with respect to time.
- In thermodynamics, there is the expansion of metal with respect to time.

Ask students to solve a problem similar to the following:

- Suppose a 6 m ladder is leaning against a wall and the bottom of the ladder is sliding away from the wall at a rate of 0.5 m/s. At what rate is the ladder sliding down the wall at the instant when the bottom of the ladder is 3 m from the wall?

The provided directions should help students discover an effective strategy required to solve related rates problems.

- Draw an appropriate diagram to represent the situation. This is the most helpful step in related rates problems since it allows students to visualize the problem.



- Assign variables to each quantity in the problem that is changing.

$t$  = time (seconds)

$x$  = the horizontal distance (metres) from the wall to the bottom of the ladder at time  $t$

$y$  = the vertical distance (metres) from the top of the ladder to the ground at time  $t$



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**General Outcome: Develop introductory calculus reasoning.**


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**Suggested Assessment Strategies**
*Paper and Pencil*

- Suppose air is being pumped into a spherical balloon so that its volume increases at a rate of  $100 \frac{\text{cm}^3}{\text{s}}$ . Ask students to determine how fast the radius of the balloon is changing when the radius is:

- (i) 2 cm
- (ii) 4 cm

(C8.1, C8.2)

- Using  $xy + y^2 = x$ , ask students to find  $\frac{dx}{dt}$  when  $y = 2$  and  $\frac{dy}{dt} = -3$ .

(C8.1, C8.2)

- Ask students to answer the following :

- (i) Two airplanes in horizontal flight cross over a town at 1 pm. One plane travels east at 300 km/h, while the other goes north at 400 km/h. At what rate is the distance between them changing at 3 pm?
- (ii) Water is flowing into a trough at a rate of  $200 \frac{\text{cm}^3}{\text{s}}$ . The trough has a length of 4 m and a cross-section is in the form of an isosceles triangle which has a height of 1 m and a top that is 2 m. At what rate is the water rising when the depth is 60 cm?

(C8.1, C8.2)

**Resources/Notes**
**Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**2.7: Related Rates**

SB: pp. 128 - 134

IG: pp. 131 - 136

PPL: SteEC\_02\_07.ppt

Video: Example 1 and 4

**Suggested Resource**

Resource Link:

[www.k12pl.nl.ca/curr/10-12/math/3208/res-link/appsderivs.html](http://www.k12pl.nl.ca/curr/10-12/math/3208/res-link/appsderivs.html)

- volume and radius of a snowball change with respect to time

## Calculus Reasoning

### Outcomes

*Students will be expected to*

**C8 Continued...**

**Achievement Indicators:**

C8.1, C8.2 *Continued*

### Elaborations – Strategies for Learning and Teaching

- List the given information and identify the unknown quantity in terms of the variables.

$$\text{length of the ladder} = 6 \text{ m}, \frac{dx}{dt} = 0.5, x = 3, \frac{dy}{dt} = ?$$

- Write an equation representing the relationship between the variables. Applying the Pythagorean theorem to the right triangle results in  $x^2 + y^2 = 6^2$ .
- Implicitly differentiate the relationship with respect to time.

$$x^2 + y^2 = 6^2$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}$$

- Substitute all given information into the equation and solve for the required rate of change.

$$\frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}$$

$$\frac{dy}{dt} = -\frac{3}{3\sqrt{3}}(0.5)$$

$$\frac{dy}{dt} = -\frac{1}{2\sqrt{3}} \text{ m/s}$$

$$\frac{dy}{dt} = -\frac{\sqrt{3}}{6} \text{ m/s}$$

Teachers should discuss the significance of the negative result given the ladder is falling down the wall.

It is important for students to wait until the equation has been differentiated to substitute information into the equation. If values are substituted too early, it can lead to an incorrect answer.

**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Paper and Pencil*

- Sand is being dumped from a conveyor belt at a rate of  $2.3 \frac{m^3}{min}$  and forms a pile in the shape of a cone whose base diameter and height are always equal. Ask students how fast the height is changing when the pile is 3 m high.

(C8.1, C8.2)

- Water is flowing into a trough at a rate of  $100 \text{ cm}^3/\text{min}$ . The trough has a length of 3 m and a cross-section is in the form of an isosceles trapezoid, where the height is 1 m, top is 2 m and the base is 50 cm. At what rate is the water rising when the depth is 25 cm?

(C8.1, C8.2)

**Resources/Notes****Authorized Resource***Single Variable Essential Calculus  
Second Edition***2.7: Related Rates**

SB: pp. 128 - 134

IG: pp. 131 - 136

PPL: SteEC\_02\_07.ppt

Video: Example 1 and 4

## Calculus Reasoning

### Outcomes

*Students will be expected to*

**C9 Use calculus techniques to solve optimization problems.**

[C, CN, ME, PS, R]

#### Achievement Indicators:

C9.1 Determine the equation of the function to be optimized in an optimization problem.

C9.2 Determine the equations of any parameters necessary in an optimization problem.

C9.3 Solve an optimization problem drawn from a variety of applications, using calculus techniques.

C9.4 Interpret the solution(s) to an optimization problem.

### Elaborations – Strategies for Learning and Teaching

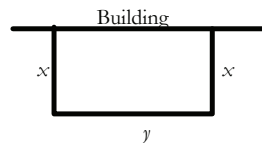
In Mathematics 2200, students solved maximum and minimum problems (RF4). They will now solve similar types of problems using derivatives, also known as optimization problems.

The following guidelines may be useful for students when solving optimization problems:

- Draw a sketch to help model the situation.
- Assign variables to non-constant quantities.
- Write any existing relationships between variables as equations.
- Write an equation that models the function to be optimized.
- Ensure the equation contains only one independent variable.
- Determine the maximum or minimum quantity.

Students could use these guidelines to solve a problem similar to:

- A holding pen is being built alongside a long building. The pen requires only three fenced sides with the building forming the fourth side. There is enough material for 60 m of fencing. What dimensions will yield a maximum area for the enclosure?



Students should notice that the two quantities being discussed are perimeter ( $P$ ) and area ( $A$ ). These can be written as  $60 = 2x + y$  and  $A = xy$ . Ensure students express the area in terms of one variable,  $A = x(60 - 2x) = 60x - 2x^2$ . Ask them to first determine the maximum area using non-calculus techniques. Since this is a quadratic function, they should make the connection between the maximum or minimum value, and the  $y$ -value of the vertex.

From a calculus perspective, students should be aware that extreme values occur at critical numbers and critical numbers occur where the first derivative equals zero or is undefined.

$$A'(x) = 60 - 4x$$

$$0 = 60 - 4x$$

$$x = 15$$

Students can verify that the critical number  $x = 15$  will produce a maximum value by either using the first or second derivative test. In this example, they can easily identify the second derivative (i.e.,  $A''(x) = -4 < 0$ ). Using substitution, students should conclude that  $x = 15$  m,  $y = 30$  m and the maximum area =  $450$  m<sup>2</sup>.

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**General Outcome: Develop introductory calculus reasoning.**


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**Suggested Assessment Strategies**
*Paper and Pencil*

- Ask students to answer the following:
  - (i) A rock is thrown upwards from a 200 m cliff. If the height of the rock above the bottom of the cliff is given by  $h(t) = -5t^2 + 20t + 200$ , determine the maximum height of the rock.
  - (ii) A box with an open top is to be constructed from a square piece of cardboard, 3 ft wide, by cutting out an identical square from each of the four corners and folding up the sides. Find the largest volume that such a box can have.
  - (iii) A Norman window has a shape of a rectangle with the top edge replaced by a semi-circle. If the window has a perimeter of 8 m, determine the dimensions of the window so that the greatest possible amount of light is admitted.
  - (iv) A container in the shape of a right cylinder with no top has a surface area of  $3\pi$  ft<sup>2</sup>. What height ( $h$ ) and base radius ( $r$ ) will maximize the volume of the cylinder?
  - (v) A cola company wishes to introduce a new can size that will contain 355 mL and be environmentally friendly. Ask students to determine the dimensions of the can such that the amount of aluminium used is a minimum.

(C9.1, C9.2, C9.3, C9.4)

**Resources/Notes**
**Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**3.5: Optimization Problems**

SB: pp. 173 - 183

IG: pp. 185 - 192

PPL: SteEC\_03\_05.ppt

Video: Example 2, 5 and 6

TEC: Module 3.5: Analyzing  
Optimization Problems



Calculus of Trigonometry

Suggested Time: 15 Hours

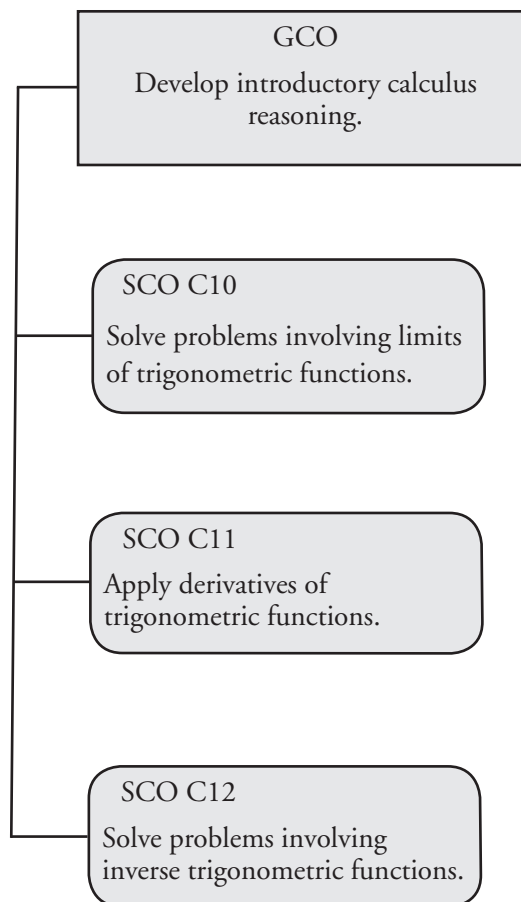
# Unit Overview

## Focus and Context

In this unit, students will evaluate limits involving trigonometric expressions and determine the derivative of expressions involving trigonometric functions. They will revisit the Chain Rule and implicit differentiation.

Students will be introduced to inverse trigonometric functions, determine the exact value of an expression involving an inverse trigonometric function, and solve problems involving the derivative of an inverse trigonometric function.

## Outcomes Framework





## SCO Continuum

Mathematics 2200	Mathematics 3200	Calculus 3208
Trigonometry	Trigonometry	Calculus
T3 Solve problems, using the cosine law and the sine law, including the ambiguous case. [C, CN, PS, R, T]	T4 Graph and analyze the trigonometric functions sine, cosine and tangent to solve problems. [CN, PS, T, V]	C10 Solve problems involving limits of trigonometric functions. [C, CN, ME, PS, R, T, V]
	T6 Prove trigonometric identities, using: <ul style="list-style-type: none"> <li>• reciprocal identities</li> <li>• quotient identities</li> <li>• Pythagorean identities</li> <li>• sum or difference identities (restricted to sine, cosine and tangent)</li> <li>• double-angle identities (restricted to sine, cosine and tangent).</li> </ul> [R, T, V]	C11 Apply derivatives of trigonometric functions. [CN, ME, R, V]
		C12 Solve problems involving inverse trigonometric functions. [CN, ME, R, V]
Relations and Functions		
	RF5 Demonstrate an understanding of inverses of relations. [C, CN, R, V]	

## Mathematical Processes

[C] Communication  
 [CN] Connections  
 [ME] Mental Mathematics and Estimation

[PS] Problem Solving  
 [R] Reasoning  
 [T] Technology  
 [V] Visualization

## Calculus Reasoning

### Outcomes

Students will be expected to

**C10 Solve problems involving limits of trigonometric functions.**

[C, CN, ME, PS, R, T, V]

### Achievement Indicator:

C10.1 Establish each of the following trigonometric limits, using informal methods:

- $\lim_{x \rightarrow 0} \sin x = 0$
- $\lim_{x \rightarrow 0} \cos x = 1$
- $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$
- $\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = 0$

### Elaborations – Strategies for Learning and Teaching

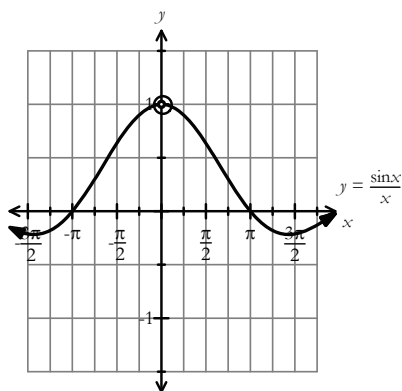
Up to this point, students have been exposed to algebraic functions when finding limits. Now students will explore limits of trigonometric functions and of inverse trigonometric functions.

In Mathematics 3200, students were introduced to radians (T1). All trigonometric functions in this unit will assume radians as input.

In Mathematics 3200, students graphed and analyzed the trigonometric functions sine, cosine and tangent (T4). They should be aware that sine and cosine functions are continuous everywhere. Inspection of the graphs of  $y = \sin x$  and  $y = \cos x$  should help students observe the  $\lim_{x \rightarrow 0} \sin x = 0$  and  $\lim_{x \rightarrow 0} \cos x = 1$ . Students can verify these results using direct substitution and make the generalization that  $\lim_{x \rightarrow a} \sin x = \sin a$  and  $\lim_{x \rightarrow a} \cos x = \cos a$ .

Students should use informal methods, such as graphing or a table of values to show that  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$  and  $\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = 0$ . It is not an expectation that students prove these limits algebraically.

Provide students with the graph of  $y = \frac{\sin x}{x}$  and ask them to examine the value of  $y$  for values of  $x$  close to 0. It may help students to organize their information in a table.



$x$	$y$
-2	0.45
-1	0.84
-0.1	0.998
-0.01	0.9999
0	
0.01	0.9999
0.1	0.998
1	0.84
2	0.45

The trend of the values of  $\frac{\sin x}{x}$  in the table appear to show that  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ . Most of the limit evaluations require students to produce this particular limit, since direct substitution will often produce  $\frac{0}{0}$ . Similarly, students should observe the graph of  $y = \frac{\cos x - 1}{x}$  to conclude that the  $\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = 0$ .

**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Observation*

- Ask students to use graphing technology to graph the function  $y = \frac{1 - \cos x}{x}$ . Ask them to use the graph and a table of values to examine the behaviour of the function as  $x$  approaches zero to determine  $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x}$ .
- (C10.1)

**Resources/Notes****Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**1.3: The Limit of a Function**

Student Book (SB): pp. 24 - 35  
Instructor's Guide (IG): pp. 19 - 34

Power Point Lecture (PPL):  
SteEC\_01\_03.ppt

Video Examples: 4

**1.4: Calculating Limits**

SB: pp. 35 - 45  
IG: pp. 35 - 40  
PPL: SteEC\_01\_04.ppt

## Calculus Reasoning

### Outcomes

Students will be expected to

C10 Continued...

#### Achievement Indicator:

C10.2 Evaluate limits involving trigonometric expressions.

### Elaborations – Strategies for Learning and Teaching

In Mathematics 3200, students worked with trigonometric identities (T6). Computation of trigonometric limits entails both algebraic manipulation and some basic trigonometric identities. A review, therefore, may be necessary.

When working with limits involving the indeterminate form  $\frac{0}{0}$ , students continue to algebraically manipulate the expression in order to evaluate the limit. They should use previous trigonometric limits (i.e.,  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ ) to help them evaluate limits such as  $\lim_{x \rightarrow 0} \frac{\sin 5x}{x}$ .

Ask students to first rewrite the function by multiplying and dividing by 5.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin 5x}{x} &= \lim_{x \rightarrow 0} \frac{\sin 5x}{x} \cdot \frac{5}{5} \\ \lim_{x \rightarrow 0} \frac{\sin 5x}{5x} \cdot 5 & \quad \text{Let } u = 5x \\ & \quad \text{As } x \rightarrow 0, \text{ then } 5x \rightarrow 0 \end{aligned}$$

Students can then use the theorem,  $\lim_{u \rightarrow 0} \frac{\sin u}{u} = 1$ , to determine the limit as follows:

$$\begin{aligned} \lim_{u \rightarrow 0} \frac{\sin u}{u} \cdot 5 \\ 5 \lim_{u \rightarrow 0} \frac{\sin u}{u} \\ 5(1) \\ 5 \end{aligned}$$

This can lead into a discussion around the general case  $\lim_{x \rightarrow 0} \frac{\sin kx}{x} = k$ , where  $k \neq 0$ .

When students evaluate a limit such as  $\lim_{x \rightarrow 0} \frac{\sin 7x}{6x}$ , it may be helpful for students to continue to use substitution where  $u = 7x$  and  $x = \frac{u}{7}$ . They can rewrite the limit as:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin 7x}{6x} & \quad \text{as } x \rightarrow 0, \text{ then } 7x \rightarrow 0, \text{ so } u \rightarrow 0 \\ &= \lim_{u \rightarrow 0} \frac{\sin u}{6 \left( \frac{u}{7} \right)} \\ &= \lim_{u \rightarrow 0} \frac{\sin u}{\left( \frac{6}{7} \right) u} \\ &= \frac{7}{6} \lim_{u \rightarrow 0} \frac{\sin u}{u} \\ &= \frac{7}{6} (1) \\ &= \frac{7}{6} \end{aligned}$$

**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Paper and Pencil*

- Ask students to evaluate the following:

(i)  $\lim_{x \rightarrow 0} \frac{\sin 5x}{3x}$

(ii)  $\lim_{x \rightarrow 0} \frac{\sin 5x}{\sin 3x}$

(iii)  $\lim_{x \rightarrow 0} \frac{\sin 4x}{\sin 6x}$

(iv)  $\lim_{t \rightarrow 0} \frac{\tan 6t}{t}$

(v)  $\lim_{\theta \rightarrow 0} \frac{\sin 3\theta \sin 5\theta}{\theta^2}$

(vi)  $\lim_{x \rightarrow 0} (x + \sin x)$

(vii)  $\lim_{x \rightarrow \frac{\pi}{4}} \frac{1 - \tan x}{\sin x - \cos x}$

(viii)  $\lim_{x \rightarrow 1} \frac{\sin(x-1)}{x^2 + x - 2}$

(C10.2)

**Resources/Notes****Authorized Resource***Single Variable Essential Calculus  
Second Edition***1.4: Calculating Limits**

SB: pp. 35 - 45

IG: pp. 35 - 40

PPL: SteEC\_01\_04.ppt

**1.6: Limits Involving Infinity**

SB: pp. 56 - 69

IG: pp. 51 - 59

PPL: SteEC\_01\_06.ppt

## Calculus Reasoning

### Outcomes

*Students will be expected to*

C10 Continued...

### Achievement Indicator:

C10.2 *Continued*

### Elaborations—Strategies for Learning and Teaching

There are a number of strategies that can be used to create an equivalent form to help evaluate trigonometric limits: factoring, rationalizing, and utilizing trigonometric identities. As students evaluate a limit such as  $\lim_{x \rightarrow 0} \frac{\sin 2x}{2x^2 + x}$ , they should first notice the limit produces an indeterminate form. Ask students to factor the denominator  $\lim_{x \rightarrow 0} \frac{\sin 2x}{x(2x+1)}$  and rewrite the limit as  $\lim_{x \rightarrow 0} \frac{\sin 2x}{x} \cdot \frac{1}{(2x+1)}$ . They should then observe that since the  $\lim_{x \rightarrow 0} \frac{\sin 2x}{x} = 2$  and  $\lim_{x \rightarrow 0} \frac{1}{(2x+1)} = 1$ , the value of  $\lim_{x \rightarrow 0} \frac{\sin 2x}{2x^2 + x}$  is 2.

In order to simplify an expression, students may have to divide both the numerator and denominator by  $x$ . When evaluating a limit such as  $\lim_{x \rightarrow 0} \frac{\sin x}{x + \tan x}$ , it may help students if the following prompts are provided:

- (i) Would it help you to replace  $\tan x$  by an expression involving  $\sin x$  and  $\cos x$ ?
- (ii) What is the expression if you multiply the numerator and denominator by  $\frac{1}{x}$ ?
- (iii) Did you find terms involving  $\frac{\sin x}{x}$ ?
- (iv) What is the  $\lim_{x \rightarrow 0} \cos x$ ?

**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Observation*

- Ask students to determine the value of each limit using three different strategies: table of values, graphically, and algebraically. Students should discuss the advantages and disadvantages of each method.

(i)  $\lim_{x \rightarrow 0} \frac{\csc x}{\cot x}$

(ii)  $\lim_{x \rightarrow 0} \frac{\sin x}{x + \sin x}$

(C10.2)

**Resources/Notes****Authorized Resource***Single Variable Essential Calculus  
Second Edition***1.4: Calculating Limits**

SB: pp. 35 - 45

IG: pp. 35 - 40

PPL: SteEC\_01\_04.ppt

**1.6: Limits Involving Infinity**

SB: pp. 56 - 69

IG: pp. 51 - 59

PPL: SteEC\_01\_06.ppt

## Calculus Reasoning

## Outcomes

Students will be expected to

**C11 Apply derivatives of trigonometric functions.**

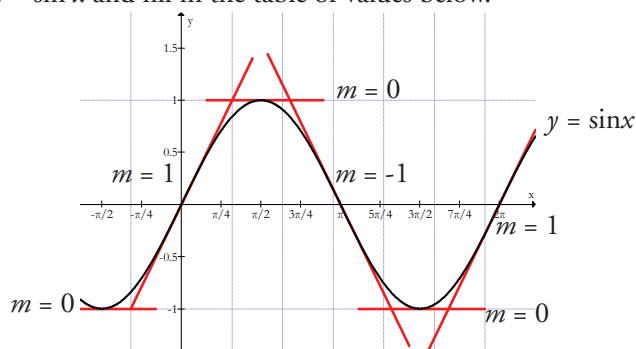
[CN, ME, R, V]

**Achievement Indicator:**

C11.1 Determine the derivatives of sine, cosine, tangent, cotangent, secant and cosecant.

## Elaborations – Strategies for Learning and Teaching

Students are aware that  $f'(x)$  is the slope of the tangent line at  $(x, f(x))$  to a function  $f(x)$ . Ask students to sketch the graph of  $f(x) = \sin x$  and fill in the table of values below.



$x$	0	$\frac{\pi}{2}$	$\pi$	$\frac{3\pi}{2}$	$2\pi$
$f'(x)$	1	0	-1	0	1

Ask students which basic periodic function has a graph that contains these points. They should observe that the derivative of the sine function appears to produce values matching those for cosine for the same  $x$  values. They can confirm this by applying the definition of the derivative to calculate  $f'(x)$  when  $f(x) = \sin x$ . To use the definition of the derivative, students may have to be reminded of the sine addition law (i.e.,  $\sin(x + h) = \sin x \cos h + \cos x \sin h$ ) that was introduced in Mathematics 3200 (T6).

$$f(x) = \sin x$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

$$f'(x) = \sin x \lim_{h \rightarrow 0} \frac{(\cos h - 1)}{h} + \cos x \lim_{h \rightarrow 0} \frac{\sin h}{h}$$

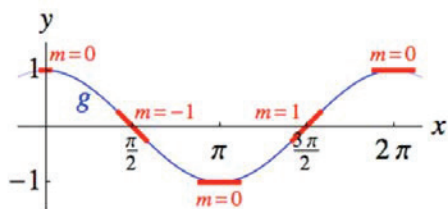
$$f'(x) = \sin x(0) + \cos x(1)$$

$$f'(x) = \cos x$$



**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Paper and Pencil*

- Ask students to graph  $f(x) = \cos x$  and determine the slope of the tangent lines at the various  $x$ -values.



They should use the definition of the derivative to prove  $f'(x) = -\sin x$ .

(C11.1)

- Ask students to verify, using the definition of the derivative, that the tangent line to the graph of  $f(x) = \sin x$  at the origin has a slope of 1 (i.e.,  $f'(0) = 1$ ).

(C11.1)

**Resources/Notes****Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**2.3: Basic Differentiation****Formulas**

SB: pp. 95 - 107

IG: pp. 102 - 108

PPL: SteEC\_02\_03.ppt

Tools for Enriching Calculus

Animations (TEC): Visual 2.3

Slope-a-Scope (Trigonometric)

**2.4: The Product and Quotient****Rules**

SB: pp. 107 - 114

IG: pp. 109 - 115

PPL: SteEC\_02\_04.ppt

## Calculus Reasoning

### Outcomes

*Students will be expected to*

C11 Continued...

### Achievement Indicator:

C11.1 *Continued*

### Elaborations – Strategies for Learning and Teaching

Using the definition of the derivative, the derivative of the cosine function can be determined. As an alternative, students could rewrite the cosine function in terms of sine and apply the Chain Rule. They should be aware that translating the sine function  $\frac{\pi}{2}$  radians yields the cosine function. Students can rewrite

$f(x) = \cos x$  as  $f(x) = \sin\left(\frac{\pi}{2} - x\right)$ . As students apply the chain rule, they should conclude the following:

$$f'(x) = \cos\left(\frac{\pi}{2} - x\right) \frac{d}{dx}\left(\frac{\pi}{2} - x\right)$$

$$f'(x) = (\sin x)(-1)$$

$$f'(x) = -\sin x$$

The trigonometric identities allow students to express the remaining trigonometric functions (i.e.,  $y = \tan x$ ,  $y = \cot x$ ,  $y = \sec x$ ,  $y = \csc x$ ), in terms of sine and cosine. They can then generate their derivatives using the Quotient Rule and Chain Rule in conjunction with the differentiation formulas for sine and cosine. Students should be aware that the Pythagorean theorem and the angle sum identities are necessary.

Guide students through the derivation of the derivatives of the trigonometric functions. When working with the trigonometric function  $y = \tan x$ , for example, students should rewrite this as  $y = \frac{\sin x}{\cos x}$ . Applying the Quotient Rule, they should write:

$$\frac{dy}{dx} = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$\frac{dy}{dx} = \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x}$$

$$\frac{dy}{dx} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$\frac{dy}{dx} = \frac{1}{\cos^2 x}$$

$$\frac{dy}{dx} = \sec^2 x$$

**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Paper and Pencil*

- Ask students to derive the derivatives of the trigonometric functions  $y = \cot x$ ,  $y = \sec x$ ,  $y = \csc x$  using the differentiation rules.

(C11.1)

**Resources/Notes****Authorized Resource***Single Variable Essential Calculus  
Second Edition***2.3: Basic Differentiation  
Formulas**

SB: pp. 95 - 107

IG: pp. 102 - 108

PPL: SteEC\_02\_03.ppt

TEC: Visual 2.3 Slope-a-Scope  
(Trigonometric)**2.4: The Product and Quotient  
Rules**

SB: pp. 107 - 114

IG: pp. 109 - 115

PPL: SteEC\_02\_04.ppt

## Calculus Reasoning

### Outcomes

Students will be expected to

C11 Continued...

#### Achievement Indicators:

C11.1 *Continued*

C11.2 *Determine the derivative of expressions involving trigonometric functions.*

### Elaborations – Strategies for Learning and Teaching

When differentiating a trigonometric function such as  $y = \csc x$ , students should write this as  $y = \frac{1}{\sin x}$  and apply the Quotient Rule. As an alternative, they could use the Power Rule and Chain Rule.

$$y = \frac{1}{\sin x}$$

$$y = (\sin x)^{-1}$$

$$\frac{dy}{dx} = -(\sin x)^{-2} \frac{d}{dx}(\sin x)$$

$$\frac{dy}{dx} = -\frac{1}{\sin^2 x}(\cos x)$$

$$\frac{dy}{dx} = -\frac{1}{\sin x} \left( \frac{\cos x}{\sin x} \right)$$

$$\frac{dy}{dx} = -\csc x \cot x$$

Students are expected to find derivatives of trigonometric expressions which require the rules of differentiation studied in previous units. Differentiating  $y = x^2 \cos x$ , for example, requires students to use the Product Rule, while differentiating  $y = \sin^5(4x + 7)$  requires students to apply the Chain Rule twice. Students should also be asked to implicitly differentiate an expression such as  $\cos x + \cos y = 1$ .

Errors generally occur when more than one application of the derivative is applied. Initially, students may confuse the format of the expression. They may incorrectly write  $\sin(5x + 1)^4$  as  $[\sin(5x + 1)]^4$ . Ensure they recognize that  $\sin^4(5x + 1)$  is equivalent to  $[\sin(5x + 1)]^4$ . Ask them to differentiate the expressions and discuss the similarities and differences in their solutions.

$$\frac{d}{dx} \sin^4(5x + 1) = 4 \sin^3(5x + 1) \cdot \cos(5x + 1) \cdot 5$$

whereas

$$\frac{d}{dx} \sin(5x + 1)^4 = \cos(5x + 1)^4 \cdot 4(5x + 1)^3 \cdot 5$$

Remind students to apply the Product Rule in situations where they have  $xy$ , for example, an expression such as

$$\frac{d}{dx} \sin(xy) = \cos(xy) \cdot (xy' + y) \cdot$$

**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Paper and Pencil*

- Ask students to determine the derivatives of the following:

(i)  $f(x) = \frac{\sin x}{x^2}$

(ii)  $f(x) = 3\sin x + \cos x - 1$

(iii)  $f(x) = (\sec x)(\csc x)$

(iv)  $y = \tan \sqrt{1-x}$

(C11.2)

- Ask students to determine  $\frac{dy}{dx}$  for  $x \cos(xy) = y \sin(3x)$ .

(C11.2)

- Ask students to differentiate  $y = \cos x$  repeatedly six times. They should explain the pattern they observe and use it to determine the 24<sup>th</sup> derivative of  $y = \cos x$ .

(C11.2)

*Observation*

- As students differentiate expressions involving trigonometric functions, teachers should look for evidence that they

(i) clearly understand the requirements of the problem,

(ii) recognize when a strategy is not appropriate,

(iii) can explain the process they are using to determine their answers, and

(iv) verify their solution(s) are correct and reasonable.

(C11.2)

*Performance*

- Provide students with questions on index cards where they have to determine the derivatives of expressions involving trigonometric functions. Vary the range of difficulty and distribute the cards randomly to students. Ask them to complete the questions and collect them once they have finished. Before the next class, organize the solutions in groups of two or three (or more if necessary) such that each group of cards has at least one problem solved incorrectly. Redistribute the cards to students and ask them to identify the errors (if any) and write the correct solution.

(C11.2)

**Resources/Notes****Authorized Resource***Single Variable Essential Calculus  
Second Edition***2.3: Basic Differentiation  
Formulas**

SB: pp. 95 - 107

IG: pp. 102 - 108

PPL: SteEC\_02\_03.ppt

**2.4: The Product and Quotient  
Rules**

SB: pp. 107 - 114

IG: pp. 109 - 115

PPL: SteEC\_02\_04.ppt

Video Examples 1 1

**2.5: The Chain Rule**

SB: pp. 114 - 122

IG: pp. 116 - 123

PPL: SteEC\_02\_05.ppt

Video Examples: 2 and 7

**2.6: Implicit Differentiation**

SB: pp. 123 - 128

IG: pp. 124 - 130

PPL: SteEC\_02\_06.ppt

## Calculus Reasoning

### Outcomes

Students will be expected to

C11 Continued...

#### Achievement Indicator:

C11.3 Solve problems involving the derivative of a trigonometric function.

### Elaborations—Strategies for Learning and Teaching

The derivative of a trigonometric function can be used to solve many problems in calculus, such as, determining the equation of a tangent line, solving related rates problems or solving maximum and minimum problems.

Work with students to find the equation of a line tangent to a function such as  $y = \frac{\cos 3x}{\sin 2x}$  at the point where  $x = \frac{\pi}{3}$ . With the aid of technology, the graph of a function and its tangent line are often easy to produce to verify student answers.

Optimization problems involving trigonometric functions should also be examined. Ask students to solve a problem such as:

- The position of a particle as it moves horizontally is described by the equation  $s = 2\sin t - \cos t$ ,  $0 \leq t \leq 2\pi$ , where  $s$  is in metres and  $t$  is the time in seconds. Find the absolute maximum and minimum displacements (i.e., the rightmost and leftmost positions) for the particle on the given interval.

Students are aware that velocity is the rate of change in displacement with respect to time.

$$v = \frac{ds}{dt} = 2 \cos t - (-\sin t)$$

$$v = \frac{ds}{dt} = 2 \cos t + \sin t$$

Remind students that the absolute maximum or minimum displacements on an interval occur when the velocity equals zero or at the endpoints of the interval. To determine the possible critical numbers, ask them to solve the trigonometric equation  $2\cos t + \sin t = 0$ .

$$2 \cos t + \sin t = 0$$

$$\sin t = -2 \cos t$$

$$\frac{\sin t}{\cos t} = -2$$

$$\tan t = -2$$

Ask students to determine the possible values of  $t$  with reference to the domain provided. The reference angle is  $\tan^{-1}(2) = 1.107$ .

$$\frac{\pi}{2} < t < \pi, t_1 = \pi - 1.107 = 2.0 \text{ sec}$$

$$\frac{3\pi}{2} < t < 2\pi, t_2 = 2\pi - 1.107 = 5.2 \text{ sec}$$

Students are not expected to solve higher order trigonometric equations where multiple simplifications and calculations are necessary (e.g.,  $\cos(3x - 2) + \sin(5x + 1) = 0$ ).

**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Paper and Pencil*

- Ask students to answer the following:
  - (i) Determine the slope of the tangent to the curve  $y = \sin x$  at the points  $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi$ . Graph the results.
  - (ii) Show that the graphs of  $y = \sin x$  and  $y = \csc x$  have horizontal tangents at  $x = \frac{\pi}{2}$ .
  - (iii) Find the equation of the tangent line of a curve  $y = \sqrt{2} \sin x$  at  $x = \frac{3\pi}{4}$ .  
(C11.3)
- Ask students to determine the equation of the tangent line of the relation  $\sin(xy - y^2) = x^2 - 1$  at the point (1,1).  
(C11.3)
- Two sides of a triangle are six and eight metres in length. If the angle between them decreases at the rate of 0.035 rad/s, ask students to determine the rate at which the area is decreasing when the angle between the sides of fixed length is  $\frac{\pi}{6}$ .  
(C11.3)
- The base of an isosceles triangle is 20 cm and the altitude is increasing at a rate of  $1 \frac{\text{cm}}{\text{min}}$ . Ask students to determine the rate which the base angle is increasing when the area is  $100 \text{ cm}^2$ .  
(C11.3)

**Resources/Notes****Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**2.4: The Product and Quotient Rules**

SB: pp. 107 - 114  
IG: pp. 109 - 115  
PPL: SteEC\_02\_04.ppt  
Video Examples: 1

**2.5: The Chain Rule**

SB: pp. 114 - 122  
IG: pp. 116 - 123  
PPL: SteEC\_02\_05.ppt  
Video Examples: 7

**2.6: Implicit Differentiation**

SB: pp. 123 - 128  
IG: pp. 124 - 130  
PPL: SteEC\_02\_06.ppt

**2.7: Related Rates**

SB: pp. 128 - 134  
IG: pp. 131 - 136  
PPL: SteEC\_02\_07.ppt  
Video Examples: 5

**3.5: Optimization Problems**

SB: pp. 173 - 183  
IG: pp. 185 - 192  
PPL: SteEC\_03\_05.ppt  
Video Examples: 5

## Calculus Reasoning

### Outcomes

Students will be expected to

C11 Continued...

### Achievement Indicator:

C11.3 Continued

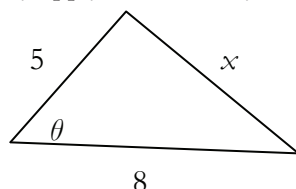
### Elaborations – Strategies for Learning and Teaching

Determining the position, students should conclude  $s(2.0) = 2.2$  m and  $s(5.2) = -2.2$  m. Ensure that they test the endpoints of the domain at  $t = 0$  and  $t = 2\pi$  to verify the maximum and minimum values. The rightmost is 2.2 m (or 2.2 m to the right) and the leftmost is -2.2 m (or 2.2 m to the left).

Problems involving related rates may involve other applications of trigonometric functions. Ask students to solve a problem such as the following:

- Two sides of a triangle measure 5 m and 8 m in length. The angle between them is increasing at a rate of  $\frac{\pi}{45} \frac{\text{rad}}{\text{sec}}$ . How fast is the length of the third side changing when the contained angle is  $\frac{\pi}{3}$ ?

Students should sketch a diagram and label the information. Students were introduced to the Law of Cosines in Mathematics 2200 (T3). As they apply this law, they should obtain the following result:



$$x^2 = 5^2 + 8^2 - 2(5)(8)\cos\theta$$

$$x^2 = 89 - 80 \cos\theta$$

Ask students to differentiate with respect to time and substitute in the given information:

$$2x \frac{dx}{dt} = -80(-\sin\theta) \cdot \frac{d\theta}{dt}$$

Given:  $\frac{d\theta}{dt} = \frac{\pi}{45} \frac{\text{rad}}{\text{sec}}$ ,  $\theta = \frac{\pi}{3}$

$$\frac{dx}{dt} = \frac{40 \sin\theta}{x} \cdot \frac{d\theta}{dt}$$

$$x^2 = 89 - 80 \cos\left(\frac{\pi}{3}\right)$$

$$x = 7 \text{ m}$$

Therefore,  $\frac{dx}{dt} = \frac{40 \sin\left(\frac{\pi}{3}\right)}{7} \cdot \frac{\pi}{45}$

$$\frac{dx}{dt} = 0.35 \frac{\text{m}}{\text{s}}$$

Students should conclude that the third side is increasing at a rate of  $0.35 \frac{\text{m}}{\text{s}}$ .



**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Paper and Pencil*

- Ask students to answer the following:
  - (i) A kite 40 m above the ground moves horizontally at the rate of  $3 \frac{\text{m}}{\text{s}}$ . At what rate is the angle between the string and the horizontal decreasing when 80 m of string has been let out?
  - (ii) The position of a particle as it moves horizontally is described by the equation  $s = 2\sin t + \sin 2t$ ,  $0 \leq t \leq 2\pi$ . If  $s$  is the position in metres and  $t$  is the time in seconds, determine the rightmost and leftmost positions.
  - (iii) A weather balloon is released and ascends straight upward at a constant rate of 30 feet per second. Angela, standing 300 feet away from the point on the ground where the balloon was released, watches the balloon through a scope.
    - (a) How fast is Angela's viewing angle increasing, in radians per second, 20 seconds after the balloon is released?
    - (b) Convert the answer in (a) to degrees per second.
    - (c) Explore why the related rate must first be determined in radians per second before the degrees per second can be determined.

(C11.3)

**Resources/Notes****Authorized Resource***Single Variable Essential Calculus  
Second Edition***2.4: The Product and Quotient Rules**

SB: pp. 107 - 114  
 IG: pp. 109 - 115  
 PPL: SteEC\_02\_04.ppt  
 Video Examples: 1

**2.5: The Chain Rule**

SB: pp. 114 - 122  
 IG: pp. 116 - 123  
 PPL: SteEC\_02\_05.ppt  
 Video Examples: 7

**2.6: Implicit Differentiation**

SB: pp. 123 - 128  
 IG: pp. 124 - 130  
 PPL: SteEC\_02\_06.ppt

**2.7: Related Rates**

SB: pp. 128 - 134  
 IG: pp. 131 - 136  
 PPL: SteEC\_02\_07.ppt  
 Video Examples: 5

**3.5: Optimization Problems**

SB: pp. 173 - 183  
 IG: pp. 185 - 192  
 PPL: SteEC\_03\_05.ppt  
 Video Examples: 5

## Calculus Reasoning

### Outcomes

*Students will be expected to*

**C12 Solve problems involving inverse trigonometric functions.**

[CN, ME, R, V]

#### Achievement Indicators:

C12.1 *Explain the relationship between the trigonometric functions and their inverses.*

C12.2 *Explain why trigonometric functions must have their domains restricted to create inverse trigonometric functions.*

### Elaborations—Strategies for Learning and Teaching

Students will represent, analyze, and determine the restrictions on the domain and range of inverse trigonometric functions. They will explore the relationship between the trigonometric functions and their inverses, both in theory and application, in a variety of problem situations.

In Mathematics 3200, students determined the inverse of a function with a focus on linear and quadratic functions and used the horizontal line test of the original function to determine if its inverse was a function. They restricted the domain of the function over a specified interval in order for its inverse to be a function (RF6). Students will now use this process to determine the inverse of trigonometric functions. It is important to note that students are not responsible for the graphs of the inverse of the reciprocal trigonometric functions (i.e.,  $\csc^{-1}x$ ,  $\sec^{-1}x$ ,  $\cot^{-1}x$ ).

Students should be able to distinguish between reciprocal trigonometric functions (secant, cosecant, cotangent) and inverse trigonometric functions. They should be aware, for example, that  $\frac{1}{\sin x} \neq \sin^{-1} x$ . The "-1" is not an exponent, it is a notation that is used to denote inverse trigonometric functions. Point out to students that if the "-1" represented an exponent it would be written as  $(\sin(x))^{-1} = \frac{1}{\sin x}$ . Expose students to another notation for inverse trigonometric functions,  $\sin^{-1}x = \arcsin(x)$ , that avoids this ambiguity.

Review with students why inverses are not always functions. After applying the horizontal line test to a quadratic graph, for example, students restrict the domain in order to come up with its inverse function. In Mathematics 3200, students explored the graphs of the primary trigonometric functions (T4). Ask them to sketch their graphs and discuss where the horizontal line test fails. They should recall that the domain must be restricted in order to have an inverse function if it fails the horizontal line test.

Since the graphs of the primary trigonometric functions are periodic, students should choose an appropriate domain so they can use all values of the range. Discuss with students how restricting the domain of the sine function on the interval  $[-\frac{\pi}{2}, \frac{\pi}{2}]$  is the best option since it yields the absolute maximum and minimum values on the sine graph while maintaining a close proximity to the origin. This should be extended to show that the restriction on the domain of the tangent graph is  $(-\frac{\pi}{2}, \frac{\pi}{2})$ . For cosine, the restriction is  $[0, \pi]$ . Students may notice that  $[-\pi, 0]$  yields the same result. Although both restrictions are correct, mathematics convention uses  $[0, \pi]$ .

**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Paper and Pencil*

- Ask students to answer the following:  
Graph the function  $y = \sin x$ . How can the domain of the sine function be restricted so that it passes the horizontal line test? Can this be done in infinitely many ways? Use the restricted sine function to define the inverse sine function. Repeat this process for the functions  $y = \cos x$  and  $y = \tan x$ .  
(C12.1, C12.2)
- Ask students to explore the origin of the arc trigonometric functions, (i.e., arcsin, arccos, arctan).  
(C12.1, C12.2)

**Resources/Notes****Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**5.6: Inverse Trigonometric  
Functions**

SB: pp. 291 - 298

IG: pp. 300 - 304

PPL: SteEC\_05\_06.ppt

Video: Example 2

**Calculus Reasoning**

**Outcomes**

Students will be expected to  
C12 Continued...

**Achievement Indicator:**

C12.3 Sketch the graph of an inverse trigonometric function.

**Elaborations – Strategies for Learning and Teaching**

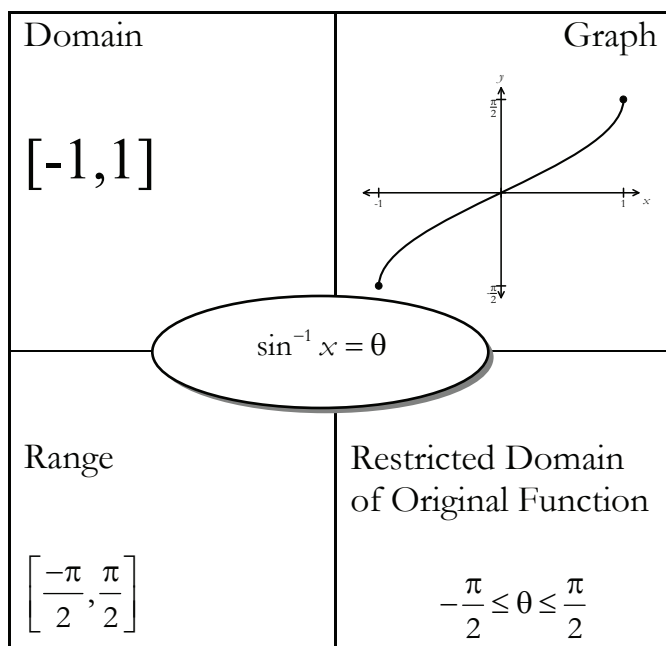
To graph  $y = \sin^{-1}x$ , for example, ask students to choose points on the graph of the restricted sine function and reverse the order of the coordinates. The points  $(-\frac{\pi}{2}, -1)$ ,  $(0, 0)$ , and  $(\frac{\pi}{2}, 1)$  on the graph of the restricted sine function become  $(-1, -\frac{\pi}{2})$ ,  $(0, 0)$  and  $(1, \frac{\pi}{2})$  on the graph of the inverse sine function. Using these three points provides students with a quick way of sketching the graph of the inverse sine function. A more accurate graph can be obtained by using graphing technology. Ask students which three points could be used when graphing  $y = \cos^{-1}x$  and  $y = \tan^{-1}x$ .

Inverse Trigonometric Function	Domain	Range	Graph
$\sin^{-1}x = \theta$ iff $\sin\theta = x$ $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$	
$\cos^{-1}x = \theta$ iff $\cos\theta = x$ $0 \leq \theta \leq \pi$	$[-1, 1]$	$[0, \pi]$	
$\tan^{-1}x = \theta$ iff $\tan\theta = x$ $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$(-\infty, \infty)$	$(-\frac{\pi}{2}, \frac{\pi}{2})$	

Although students are not responsible for the graphs of the inverse of the reciprocal trigonometric functions (i.e.,  $\csc^{-1}x$ ,  $\sec^{-1}x$ ,  $\cot^{-1}x$ ), they will be expected to derive and utilize the derivatives of the reciprocal trigonometric functions later in this course.

**General Outcome: Develop introductory calculus reasoning.**
**Suggested Assessment Strategies**
*Performance*

- Ask students to work in pairs to complete the following inverse trigonometric puzzle. A sample is shown below.



(C12.1, C12.2, C12.3)

- Ask students to sketch the graphs of the inverse trigonometric functions  $y = \sin^{-1}x$ ,  $y = \cos^{-1}x$  and  $y = \tan^{-1}x$ . They should then use graphing technology to verify their results. As teachers observe students' work, ensure they discuss the following characteristics:
  - The graph of  $\sin^{-1}x$  is a reflection of the graph of  $\sin x$  (restricted to  $[-\frac{\pi}{2}, \frac{\pi}{2}]$ ) about the line  $y = x$ .
  - The graph of  $\cos^{-1}x$  is a reflection of the graph of  $\cos x$  (restricted to  $[0, \pi]$ ) about the line  $y = x$ .
  - The graph of  $\tan^{-1}x$  is a reflection of the graph of  $\tan x$  (restricted to  $(-\frac{\pi}{2}, \frac{\pi}{2})$ ) about the line  $y = x$ .
  - The domain of a function and the range of its inverse are the same.
  - The vertical asymptotes of the tangent function become horizontal asymptotes for the inverse function.

(C12.1, C12.2, C12.3)

**Resources/Notes**
**Authorized Resource**
*Single Variable Essential Calculus  
Second Edition*
**5.6: Inverse Trigonometric  
Functions**

SB: pp. 291 - 298

IG: pp. 300 - 304

PPL: SteEC\_05\_06.ppt

Video: Example 2

## Calculus Reasoning

### Outcomes

Students will be expected to

C12 Continued...

### Achievement Indicator:

C12.4 Determine the exact value of an expression involving an inverse trigonometric function.

### Elaborations – Strategies for Learning and Teaching

When determining the exact value of an expression involving an inverse trigonometric function, ask students to rewrite the inverse trigonometric function as its equivalent trigonometric function. When determining the exact value of  $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ , for example, they should be asking themselves "The sine of what angle results in  $\frac{\sqrt{3}}{2}$ ?" Encourage students to write this as  $\sin x = \frac{\sqrt{3}}{2}$ . They might initially predict that there are many solutions since  $x = \frac{\pi}{3} + 2\pi k$  and  $x = \frac{2\pi}{3} + 2\pi k$ , where  $k$  is an integer. Remind students of the restricted domains of the inverse trigonometric functions. Discuss with them that the only angle that falls in the range of  $\arcsin x$  is  $\frac{\pi}{3}$ . Ask students how this answer would change if the expression was  $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ .

Students should also be exposed to problems that involve the composition of functions such as  $\sin^{-1}\left(\sin\left(\frac{\pi}{4}\right)\right)$ ,  $\cos\left(\cos^{-1}\left(\frac{1}{2}\right)\right)$ ,  $\sin^{-1}\left(\frac{\cos 7\pi}{3}\right)$ . Ask students to evaluate an expression such as  $\sin^{-1}\left(\sin\frac{\pi}{6}\right)$  and verify the answer is  $\frac{\pi}{6}$ . When working with  $\sin^{-1}(\sin x)$ , students may incorrectly assume the answer is always  $x$ . Ensure students recognize that this is only true when the value of  $x$  falls within the restricted domain  $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ . The expression  $\sin^{-1}\left(\sin\frac{7\pi}{6}\right)$ , for example, simplifies to  $-\frac{\pi}{6}$ . Introduce students to the general properties of inverse functions:

- $\sin^{-1}(\sin x) = x$  for  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$
- $\sin(\sin^{-1}x) = x$  for  $-1 \leq x \leq 1$

Similar discussions should take place concerning  $\cos^{-1}(\cos x)$ ,  $\cos(\cos^{-1}x)$ ,  $\tan^{-1}(\tan x)$  and  $\tan(\tan^{-1}x)$ .

**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Paper and Pencil*

- Ask students to answer the following:
  - (i) Given  $\theta = \sin^{-1}\left(\frac{8}{17}\right)$ , find the values of all six trigonometric functions at  $\theta$ .
  - (ii) If the point  $(-3, 4)$  is on the terminal arm of  $\theta$ , find the values of all six trigonometric functions at  $\theta$ .
  - (iii) Evaluate  $\tan(\cos^{-1}\left(\frac{7}{11}\right))$ .
  - (iv) Find the exact value of  $\sin(2 \tan^{-1} \sqrt{2})$ .
  - (v) Find the exact value of  $\cos(\tan^{-1}2 + \tan^{-1}3)$ .

(C12.4)

**Resources/Notes****Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**5.6: Inverse Trigonometric Functions**

SB: pp. 291 - 298

IG: pp. 300 - 304

PPL: SteEC\_05\_06.ppt

Video: Example 2

## Calculus Reasoning

### Outcomes

Students will be expected to

C12 Continued...

### Achievement Indicator:

C12.5 Derive the inverse trigonometric derivatives.

### Elaborations – Strategies for Learning and Teaching

It is important to guide students through the process of developing the derivatives of the inverse trigonometric functions using the Pythagorean theorem and implicit differentiation. It is not the intent for students to reproduce the derivations for assessment purposes.

Teachers could begin determining the derivative of the inverse sine function and the derivatives of the other inverse trigonometric functions in a similar manner.

Students are aware that the inverse sine function  $y = \sin^{-1}x$ ,  $-1 \leq x \leq 1$ ,  $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$  was created by restricting the domain of the function  $y = \sin x$  to  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ . Remind them that  $y = \sin^{-1}x$  can be written as  $x = \sin y$ . Ask students to follow the directions to help them determine the derivative of the inverse sine function:

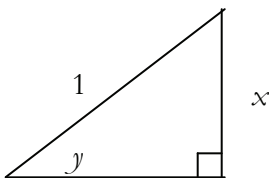
- Differentiate  $x = \sin y$  implicitly

$$\frac{d}{dx}(x) = \frac{d}{dx}(\sin y)$$

$$1 = \cos y \frac{dy}{dx}$$

$$\sec y = \frac{dy}{dx}$$

- Construct a reference triangle to record the relationship between  $x$  and  $y$  where  $\sin y = \frac{x}{1}$ . Remind students that since  $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ , the reference triangle can be drawn in quadrant I or IV. Students should, however, result in the same answer. The length of the hypotenuse is always positive.



- Ask students to determine an expression for the missing side of the right triangle.
- Use substitution to determine the ratio for  $\sec y$ :  $\sec y = \frac{1}{\sqrt{1-x^2}}$
- Write an expression for the derivative:  $\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$



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**General Outcome: Develop introductory calculus reasoning.**

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**Suggested Assessment Strategies**

*Paper and Pencil*

- Ask students to determine  $y'$  given that  $y = \arcsin(5x)$ .

(C12.5, C12.6)

**Resources/Notes**

**Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**5.6: Inverse Trigonometric  
Functions**

SB: pp. 291 - 298

IG: pp. 300 - 304

PPL: SteEC\_05\_06.ppt

Video: Example 2

**Calculus Reasoning**

**Outcomes**

*Students will be expected to*

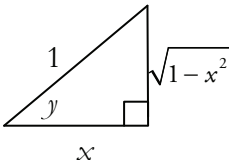
C12 Continued...

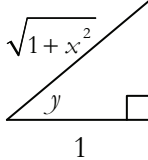
**Achievement Indicator:**

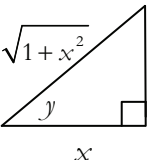
C12.5 *Continued*

**Elaborations – Strategies for Learning and Teaching**

Discuss with students that implicit differentiation can be used to develop the derivatives of  $y = \cos^{-1}x$  and  $y = \tan^{-1}x$ . Ensure that they observe how the labelling of the triangle is dependent on the function being differentiated.

- If  $y = \cos^{-1}x$ , then  $x = \cos y$ 

where  $0 \leq y \leq \pi$

- If  $y = \tan^{-1}x$ , then  $x = \tan y$ 

 $x$  where  $-\frac{\pi}{2} < y < \frac{\pi}{2}$

- If  $y = \cot^{-1}x$ , then  $x = \cot y$ 

 $1$  where  $0 < y < \pi$

Encourage students to verify the derivative formulas by applying the Pythagorean identities. Ask students to refer back to the arcsine function so they can verify the derivative result. Ask students to begin by writing  $y = \sin^{-1}x$  as  $x = \sin y$ . Guide students using the following directions:

- Differentiate both sides of the equation resulting in  $\frac{dy}{dx} = \frac{1}{\cos y}$ .
- Use substitution where  $\cos^2 y = 1 - \sin^2 y$  and  $x = \sin y$  to conclude that  $\cos y = \sqrt{1 - x^2}$ .
- Rewrite as  $\frac{dy}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ .

Although the majority of problems involve the derivatives of  $y = \sin^{-1}x$ ,  $y = \cos^{-1}x$  and  $y = \tan^{-1}x$ , students should have an opportunity to explore the derivatives of the remaining inverse trigonometric functions.

Students should use graphing technology to explore the graphs of  $y = \cot x$ ,  $y = \sec x$  and  $y = \csc x$ . With teacher guidance, observe the restrictions on each to produce the reciprocal inverse trigonometric functions  $y = \cot^{-1}x$ ,  $y = \sec^{-1}x$  and  $y = \csc^{-1}x$ .

**General Outcome: Develop introductory calculus reasoning.**
**Suggested Assessment Strategies**
*Paper and Pencil*

- Ask students to identify and correct any errors in the given solution.

$$y = \tan^{-1}(3x)$$

$$\frac{dy}{dx} = \frac{1}{1+(3x)^2}$$

$$\frac{dy}{dx} = \frac{1}{1+9x^2}$$

(C12.6)

- Ask students to find  $f'(x)$  for each of the following:

(i)  $f(x) = \arcsin x + \arccos x$

(ii)  $f(x) = \cos^{-1} \sqrt{2x-1}$

(iii)  $f(x) = \frac{\sin^{-1} x}{\cos^{-1} x}$

(iv)  $f(x) = (\tan^{-1} x)^{-1}$

(v)  $f(x) = \cos^{-1} \left( \frac{b + a \cos x}{a + b \cos x} \right)$

(C12.6)

- Ask students to differentiate  $y^2 \sin x = \tan^{-1}(x) - y$  with respect to  $x$ .

(C12.6)

- Ask students to summarize the derivatives of the inverse trigonometric functions in a table and identify any patterns they observe.

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$	$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$
$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{ x \sqrt{x^2-1}}$	$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{ x \sqrt{x^2-1}}$

(C12.5)

**Resources/Notes**
**Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**5.6: Inverse Trigonometric  
Functions**

SB: pp. 291 - 298

IG: pp. 300 - 304

PPL: SteEC\_05\_06.ppt

Video: Example 2

**Calculus Reasoning**

**Outcomes**

*Students will be expected to*  
**C12 Continued...**

**Achievement Indicators:**

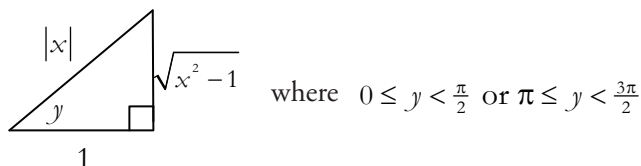
C12.5 *Continued*

C12.6 *Determine the derivative of an inverse trigonometric function.*

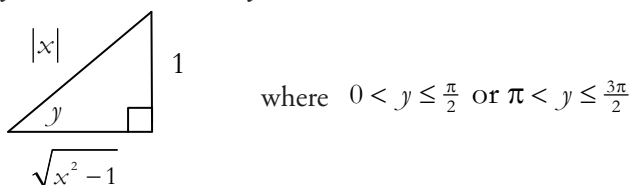
C12.7 *Solve problems involving the derivative of an inverse trigonometric function.*

**Elaborations – Strategies for Learning and Teaching**

- If  $y = \sec^{-1}x$ , then  $x = \sec y$



- If  $y = \csc^{-1}x$ , then  $x = \csc y$



Teachers should draw attention to the derivatives of  $y = \sec^{-1}x$  and  $y = \csc^{-1}x$  that involve  $|x|$ . Remind students that  $x$  represents the length of the hypotenuse, therefore, its measure is always positive.

Students should notice the derivatives of the inverse cofunctions are the negative of the derivatives of the corresponding inverse functions.

Many problems are composite in nature. Ask students to differentiate expressions similar to  $\sin^{-1}(1 - x^2)$ ,  $\tan^{-1}(\sin x)$  and  $x^2 \cos^{-1}(\frac{2}{x})$ .

Students should solve problems that are modeled using inverse trigonometric functions. They will also continue to determine the equation of a tangent line to a curve represented by an inverse trigonometric function.

**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Paper and Pencil*

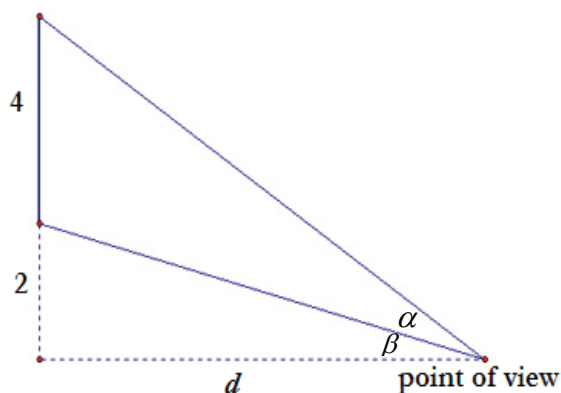
- Ask students to write the equation of the tangent line to:

(i)  $f(x) = x \sin^{-1}\left(\frac{x}{4}\right) + \sqrt{16 - x^2}$  at  $x = 2$ .

(ii)  $y = \arccos\left(\frac{x}{2}\right)$  at the point  $\left(1, \frac{\pi}{3}\right)$ .

(iii)  $y = \tan^{-1}x$  at the point  $\left(1, \frac{\pi}{4}\right)$ .

- A particle moves horizontally so that its displacement in metres after  $t$  seconds is given by  $s(t) = \tan^{-1}(\sqrt{t-1})$ . Ask students to determine the velocity of the particle when  $t = 10$  sec.
- A 4 by 4 picture hangs on a wall such that its bottom edge is 2 ft above your eye level. Ask students to determine how far from the wall should the observer stand to get the best view (i.e., How far back from the picture should the observer stand, directly in front of the picture, in order to view the picture under the maximum angle  $\alpha$ )?



(C12.6, C12.7)

**Resources/Notes****Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**5.6: Inverse Trigonometric  
Functions**

SB: pp. 291 - 298

IG: pp. 300 - 304

PPL: SteEC\_05\_06.ppt



Calculus of Exponential,  
Logarithmic, and  
Hyperbolic Trigonometric Functions

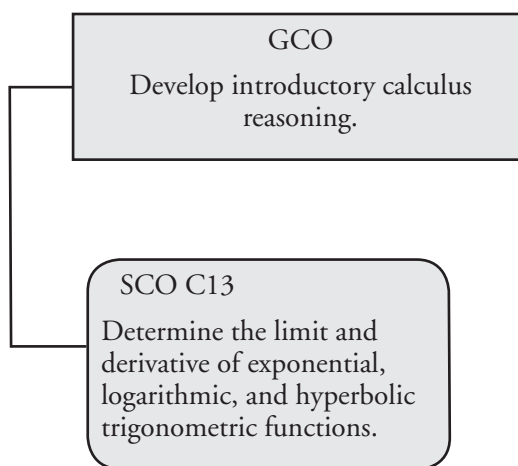
Suggested Time: 12 Hours

# Unit Overview

## Focus and Context

In this unit, students will determine the limit and derivative of exponential and logarithmic functions. They will be exposed to logarithmic differentiation to simplify a complicated derivative involving products, quotients, or powers using the rules of logarithms. Students will also be exposed to the derivatives of hyperbolic trigonometric functions which are based on natural base  $e$  and which illustrate connections between trigonometry and exponential functions.

## Outcomes Framework





## SCO Continuum

Mathematics 2200	Mathematics 3200	Calculus 3208
Relations and Functions		Calculus
not addressed	<p>RF7 Demonstrate an understanding of the product, quotient and power rules of logarithms.</p> <p>[C, CN, ME, R, T]</p> <p>RF8 Graph and analyze exponential and logarithmic functions.</p> <p>[C, CN, T, V]</p>	<p>C13 Determine the limit and derivative of exponential, logarithmic, and hyperbolic trigonometric functions.</p> <p>[CN, ME, R, V]</p>

## Mathematical Processes

[C] Communication  
 [CN] Connections  
 [ME] Mental Mathematics and Estimation

[PS] Problem Solving  
 [R] Reasoning  
 [T] Technology  
 [V] Visualization

## Calculus Reasoning

### Outcomes

*Students will be expected to*

**C13 Determine the limit and derivative of exponential, logarithmic, and hyperbolic trigonometric functions.**

[CN, ME, R, V]

### Achievement Indicator:

C13.1 *Graph and analyze exponential and logarithmic functions  $y = e^x$  and  $y = \ln x$ .*

### Elaborations—Strategies for Learning and Teaching

In this unit, students will recognize that the natural logarithmic function  $f(x) = \log_e x$ , also written as  $f(x) = \ln x$ , is the inverse of the exponential function  $f(x) = e^x$ . They will extend their understanding of differential calculus by exploring and applying the derivatives of exponential and logarithmic functions.

Students will use the technique of logarithmic differentiation to differentiate more complicated functions using the properties of the logarithm.

In Mathematics 3200, students graphed and analyzed exponential and logarithmic functions (RF9). They were not exposed to the natural logarithm. Students should now be introduced to the natural logarithm and the number  $e$ .

Introduce the number  $e$  as an irrational number, similar in nature to  $\pi$ . Its non-terminating, non-repeating value is  $e = 2.718\ 28182 \dots$ . This value can be defined by calculating  $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$ . The intent here is for students to informally look at the value for  $e$ . They are not expected to perform a mathematical proof. Ask students to use their calculator to confirm that for  $n = 10, 100, 1000, 10,000, 100,000, 1,000,000$  the values of  $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$  are rounding to 2.59, 2.70, 2.717, 2.718, 2.71827, 2.718280. They should notice that, as  $n$  approaches infinity,  $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$  approaches a definite limit.

Teachers should define the natural logarithm function  $\ln x$  as the inverse of the exponential function  $y = e^x$ . Students are familiar with inverses in Mathematics 3200 (RF8) and should make the connection that the graph of  $y = \ln x$  (i.e.,  $\log_e x$ ) is a reflection of the graph of  $y = e^x$  about the line  $y = x$ .

**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Interview*

- Ask students to evaluate  $\lim_{x \rightarrow -\infty} e^x$  using both a table of values and a graph.

Use the following questions to help them make the connection to the horizontal asymptote of  $y = e^x$  and the vertical asymptote of  $y = \ln x$ . Ask students the following questions:

- What aspect of the graph of  $y = e^x$  does  $\lim_{x \rightarrow -\infty} e^x$  reveal?
- How can you use the table of values and the graph of  $y = e^x$  to determine the  $\lim_{x \rightarrow 0} \ln x$ ?
- What connection can you make to the limit and asymptotes?

(C13.1)

**Resources/Notes****Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**5.2: The Natural Logarithmic Function**

Student Book (SB): pp. 261 - 269  
Instructor's Guide (IG): pp. 276 - 283

Power Point Lecture (PPL):  
SteEC\_05\_02.ppt

Video Examples: 4

**5.3: The Natural Exponential Function**

SB: pp. 270 - 276  
IG: pp. 284 - 286

PL: SteEC\_05\_03.ppt

**Note:**

Integration of logarithms and exponentials is not an outcome of this course.

## Calculus Reasoning

### Outcomes

Students will be expected to

C13 Continued...

#### Achievement Indicators:

C13.2 Establish the exponential limit  $\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$  using informal methods.

C13.3 Derive the derivatives of the exponential functions  $e^x$  and  $a^x$ , and the logarithmic functions  $\ln x$  and  $\log_a x$ .

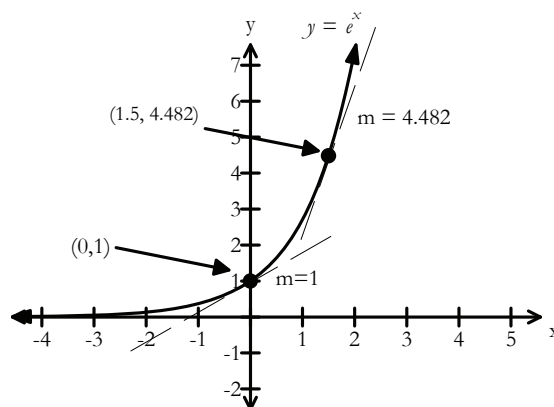
### Elaborations – Strategies for Learning and Teaching

In order to compute the derivative of the exponential function  $y = e^x$ , students must first establish the value of  $\lim_{h \rightarrow 0} \frac{e^h - 1}{h}$ . Ask students to choose values of  $h$  that are close to 0 to determine the value of the limit. Provide students with the sketch of its graph to verify the limit is 1. Ask students what happens when  $h = 0$ .

$h$	$\frac{e^h - 1}{h}$
-0.1	0.9516258196
-0.01	0.9950166251
-0.001	0.9995001666
0	?
0.1	1.051709181
0.01	1.005016708
0.001	1.000500167

Guide students through the derivations of the derivative of the exponential and logarithmic functions,  $y = e^x$ ,  $y = \ln x$ ,  $y = \log_a x$ , and  $y = a^x$ . Inform students that the derivative of  $y = \ln x$  is based on the derivative  $y = e^x$ , and the derivative of  $y = \log_a x$  is based on the derivative of  $y = \ln x$ . Students should be able to prove that  $\frac{d}{dx} e^x = e^x$  using the definition of derivative.

Using graphing technology, ask students to graph the exponential function  $y = e^x$  and its derivative at various  $x$ -values.



They should observe that the slope of the tangent to the curve at  $y = e^x$  is equal to the  $y$ -coordinate of the point of tangency.

**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Interview*

- Ask students to select three  $y$ -values on the graph of  $y = e^x$ . Ask them to find the slope of the tangent to the curve at these points. Ask them what they notice.

(C13.3)

**Resources/Notes****Authorized Resource***Single Variable Essential Calculus  
Second Edition***5.3: The Natural Exponential  
Function**

SB: pp. 270 - 276

IG: pp. 284 - 286

PPL: SteEC\_05\_03.ppt

**Note:**Achievement Indicator C13.2 is  
not addressed in the student book.

## Calculus Reasoning

### Outcomes

Students will be expected to

C13 Continued...

#### Achievement Indicators:

C13.3 Continued

C13.4 Determine the derivative of an exponential function.

C13.5 Solve problems involving the derivative of an exponential or a logarithmic function.

### Elaborations—Strategies for Learning and Teaching

Ask students to verify that the derivative of  $y = e^x$  is itself by computing the derivative using the definition of the derivative and the fact that the  $\lim_{h \rightarrow 0} \frac{e^h - 1}{h}$  is 1.

$$f(x) = e^x$$

$$\frac{df}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{df}{dx} = \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h}$$

$$\frac{df}{dx} = \lim_{h \rightarrow 0} \frac{e^x e^h - e^x}{h}$$

$$\frac{df}{dx} = \lim_{h \rightarrow 0} \frac{e^x (e^h - 1)}{h}$$

$$\frac{df}{dx} = e^x \lim_{h \rightarrow 0} \frac{(e^h - 1)}{h}$$

$$\frac{df}{dx} = e^x (1) = e^x$$

Remind students that the chain rule is applied when computing the derivative of a composite function. This is expressed as  $\frac{d}{dx}(e^u) = e^u \frac{du}{dx}$  where  $u$  is a function of  $x$ .

It may be beneficial for students to work with derivatives involving  $e^x$  before finding the derivatives of logarithmic functions. Ask students to determine the derivative of functions such as:

$$(i) \quad y = e^{3x-7x^2} \quad (ii) \quad y = e^{\sec x}$$

$$(iii) \quad y = e^{\arctan x} \quad (iv) \quad y = \frac{e^{2x}}{\sin(e^x)}$$

Derivatives of exponential functions can be used to solve problems which may involve population growth, investments, equation of a tangent line, maximum or minimum points, etc. Ask students to solve problems such as the following:

- Determine the equation of the line tangent to the curve  $y = 2e^x$  at  $x = \ln 3$ .
- Michelle just bought a motorcycle for \$12 000. The value of the motorcycle depreciates with time. The value can be modelled by the function  $V(t) = 12000e^{-\frac{t}{4}}$  where  $V$  is the value of the motorcycle after  $t$  years. At what rate is the value of the motorcycle depreciating the instant Michelle drives it off the lot? Determine the rate of depreciation at 5.5 years. What do you notice about the rate of depreciation over time?

**General Outcome: Develop introductory calculus reasoning.**
**Suggested Assessment Strategies**
*Paper and Pencil*

- Ask students to differentiate the following:

(i)  $y = e^x (x^2)$

(ii)  $y = e^{\sin x}$

(iii)  $y = \sin(e^x)$

(iv)  $y = e^{x(\sin x)}$

(v)  $y = e^{e^x}$

(C13.4)

- Ask students to compute the derivative of  $y = e^x \cdot e^{x^2}$  in two different ways.

(C13.4)

- Ask students to show that  $\frac{d}{dx} \left( \frac{e^x + e^{-x}}{e^x - e^{-x}} \right) = \frac{-4}{(e^x - e^{-x})^2}$ .

(C13.4)

- Ask students to answer the following:

- (i) Find the absolute minimum of the function

$$f(x) = \frac{e^x}{x}, x > 0.$$

- (ii) The spread of the flu virus in a certain school is modeled by the equation  $f(t) = \frac{200}{1+e^{3-t}}$  where  $f(t)$  is the total number of students infected  $t$  days after the flu was first noticed.

- Estimate the initial number of students infected with the flu.
- How fast is the flu spreading after 4 days (i.e., the rate at which the flu is spreading)?
- Using graphing technology, determine when the flu spread is at its maximum rate. What is this rate?

(C13.4, C13.5)

**Resources/Notes**
**Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**5.3: The Natural Exponential  
Function**

SB: pp. 270 - 276

IG: pp. 284 - 286

PPL: SteEC\_05\_03.ppt

## Calculus Reasoning

### Outcomes

Students will be expected to

C13 Continued...

**Achievement Indicators:**

C13.3 Continued

C13.6 Determine the derivative of a logarithmic function.

C13.5 Continued

### Elaborations – Strategies for Learning and Teaching

Students should be aware that if  $y = \ln x$  then  $e^y = x$ . Ask students to implicitly differentiate the exponential form and apply the chain rule to show that  $\frac{d}{dx}(\ln x) = \frac{1}{x}$ .

$$e^y = x$$

$$e^y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{e^y}$$

$$\frac{dy}{dx} = \frac{1}{x}$$

Students should apply the Chain Rule when computing the derivative of a composite function. This is expressed as  $\frac{d}{dx}(\ln u) = \frac{1}{u} \cdot \frac{du}{dx}$ , where  $u$  is a function of  $x$ .

Ask students to differentiate a function such as  $y = \ln\left(\frac{x}{3x-4}\right)$ .

$$y = \ln\left(\frac{x}{3x-4}\right)$$

$$y' = \frac{1}{\left(\frac{x}{3x-4}\right)} \cdot \left(\frac{(3x-4)(1) - (x)(3)}{(3x-4)^2}\right)$$

$$y' = \frac{3x-4}{x} \cdot \left(\frac{-4}{(3x-4)^2}\right)$$

$$y' = \frac{-4}{x(3x-4)}$$

In Mathematics 3200, students were introduced to the properties of logarithms (RF7). Point out to students that the natural logarithm is the logarithm of base  $e$ , therefore, the properties of logarithms for all bases apply.

$$\ln(a \cdot b) = \ln a + \ln b$$

$$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$$

$$\ln(a^k) = k \ln a$$

Students should continue to solve problems involving the derivative of logarithmic functions. Ask students to determine, for example, if the function  $y = (x-6)\ln x$  is increasing or decreasing at  $x = 1$  and then at  $x = 5$ .



**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Paper and Pencil*

- Ask students to differentiate the following:

(i)  $y = x^2 \ln x$

(ii)  $y = \ln(\cos x)$

(iii)  $y = \ln(\sin x^2)$

(iv)  $y = \ln(\ln x)$

(v)  $y = \ln(3x - 4)^2$

(vi)  $y = \ln(x)^3$

(C13.6)

- Ask students to determine the equation of the tangent to the curve,  $y = x \ln x$  at  $x = 1$ . Ask them how this answer would change if the equation was  $y = x(\log x)$  at  $x = 1$ .

(C13.5, C13.6)

**Resources/Notes****Authorized Resource***Single Variable Essential Calculus  
Second Edition***5.2: The Natural Logarithmic  
Function**

SB: pp. 261 - 269

IG: pp. 276 - 283

PPL: SteEC\_05\_02.ppt

Video Examples: 5, 6 and 9

**Calculus Reasoning**
**Outcomes**

Students will be expected to

C13 Continued...

**Achievement Indicators:**

C13.6 Continued

C13.3 Continued

**Elaborations – Strategies for Learning and Teaching**

Using the previous function, ask students to use the properties of logarithms to first simplify the equation and then differentiate.

$$y = \ln\left(\frac{x}{3x-4}\right)$$

$$y = \ln(x) - \ln(3x - 4)$$

$$y' = \frac{1}{x} - \frac{1}{3x-4} \quad (3)$$

$$y' = \frac{1}{x} - \frac{3}{3x-4}$$

This simplifies to  $y' = \frac{-4}{x(3x-4)}$ . Ask students to compare the two methods and discuss which method they prefer and why.

To determine the derivative of logarithms with bases other than  $e$ , students should change the logarithm into a natural logarithm by using the change of base formula.

$$y = \log_a x = \frac{\ln x}{\ln a}$$

$$\frac{dy}{dx} = \frac{1}{\ln a} \cdot \frac{d}{dx}(\ln x)$$

$$\frac{dy}{dx} = \frac{1}{\ln a} \cdot \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{1}{x \ln a}$$

As students conclude that  $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$ , they should also apply the Chain Rule when computing the derivative of a composite function.

This is expressed as  $\frac{d}{dx}(\log_a u) = \frac{1}{u \ln a} \cdot \frac{du}{dx}$ , where  $u$  is a function of  $x$ .

The result of  $\frac{d}{dx}(\log_a u) = \frac{1}{u \ln a} \cdot \frac{du}{dx}$  should help students determine the derivative of  $y = a^x$ . Ask students to rewrite  $y = a^x$  in logarithmic form  $x = \log_a y$ .

$$x = \log_a y$$

$$1 = \frac{1}{y \ln a} \frac{dy}{dx}$$

$$\frac{dy}{dx} = y \ln a$$

$$\frac{dy}{dx} = a^x \ln a$$

Students should conclude that  $\frac{d}{dx}(a^x) = a^x \ln a$ . Applying the chain rule, results in the  $\frac{d}{dx}(a^u) = a^u \ln a \cdot \frac{du}{dx}$ , where  $u$  is a function of  $x$ .

**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Paper and Pencil*

- Ask students to differentiate the following:

(i)  $y = \log(x + 1)$

(ii)  $y = \log_2(x^3)$

(iii)  $y = 3^{\sin x}$

(iv)  $y = \tan(3^{x^2})$

(C13.4, C13.6)

- Ask students to use the laws of logarithms to simplify and then differentiate the following:

(i)  $y = \ln [(x + 1)^2(4x - 1)^3]$

(ii)  $y = \log_3 \sqrt{x^3 + x^2}$

(iii)  $y = \log\left(\frac{x+1}{x-1}\right)$

(C13.6)

**Resources/Notes****Authorized Resource***Single Variable Essential Calculus  
Second Edition***5.2: The Natural Logarithmic  
Function**

SB: pp. 261 - 269

IG: pp. 276 - 283

PPL: SteEC\_05\_02.ppt

Video Examples: 3, 5, 6 and 9

**5.4: General Logarithmic and  
Exponential Functions**

SB: pp. 276 - 283

IG: pp. 287 - 293

PPL: SteEC\_05\_04.ppt

Video Examples: 3 and 5

## Calculus Reasoning

### Outcomes

Students will be expected to

C13 Continued...

#### Achievement Indicator:

C13.7 Determine the derivative of a function using logarithmic differentiation.

### Elaborations—Strategies for Learning and Teaching

Logarithmic differentiation is a technique where students take the logarithm of both sides of the function, rearrange terms using the logarithm laws, and then differentiate both sides implicitly. While it is acceptable to use logarithms with any given base, it is simplest to use the natural logarithm. Discuss with students the advantages of using this technique:

- To simplify a complicated derivative involving products, quotients and powers using the rules of logarithms.
- To deal with functions that have variables in the base as well as in the exponent.

Use the following prompts to guide students as they differentiate a function such as  $y = \left(\frac{x+2}{2x+1}\right)^3$  using logarithmic differentiation:

- Take the natural logarithm of both sides and apply the logarithm properties.

$$y = \left(\frac{x+2}{2x+1}\right)^3$$

$$\ln y = \ln \left(\frac{x+2}{2x+1}\right)^3$$

$$\ln y = 3[\ln(x+2) - \ln(2x+1)]$$

- Use implicit differentiation and logarithm formula derivatives.

$$\frac{1}{y} y' = 3\left[\frac{1}{x+2} - \frac{2}{2x+1}\right]$$

$$y' = 3\left(\frac{x+2}{2x+1}\right)^3 \left[\frac{1}{x+2} - \frac{2}{2x+1}\right]$$

Discuss with students that logarithmic differentiation is useful in this equation because, without it, they would have to apply the power rule, chain rule and quotient rule.

Logarithmic differentiation techniques offer a more efficient method to differentiate multifaceted functions. Logarithmic differentiation is also the only way to find the derivative of functions such as  $y = x^x$ .

$$y = x^x$$

$$\ln y = \ln x^x$$

$$\ln y = x \ln x$$

$$\frac{1}{y} \cdot y' = 1 \cdot \ln x + x \cdot \frac{1}{x}$$

$$y' = y \cdot (\ln x + 1)$$

$$y' = x^x \cdot (\ln x + 1)$$

Students may try to use the power rule and incorrectly write  $y' = x \cdot x^{x-1}$ . Teachers should emphasize that the power rule can only be used when the exponent is a constant real number.

**General Outcome: Develop introductory calculus reasoning.**
**Suggested Assessment Strategies**
*Paper and Pencil*

- Ask students to differentiate the following:

$$(i) \quad y = \frac{(2x+1)^3(x^3+x)^4}{(x^2+1)^2}$$

$$(ii) \quad y = \frac{x^{\frac{2}{3}}\sqrt{x^2+2}}{(x^3(6x+1))^3}$$

$$(iii) \quad y = x^{\cos x}$$

(C13.7)

- Ask students to show the following:

$$\frac{d}{dx}(\cos x)^{\tan x} = (\cos x)^{\tan x} [\sec^2 x \ln(\cos x) - \tan^2 x]$$

(C13.7)

**Resources/Notes**
**Authorized Resource**
*Single Variable Essential Calculus  
Second Edition*
**5.2: The Natural Logarithmic  
Function**

SB: pp. 261 - 269

IG: pp. 276 - 283

PPL: SteEC\_05\_02.ppt

**5.4: General Logarithmic and  
Exponential Functions**

SB: pp. 276 - 283

IG: pp. 287 - 293

PPL: SteEC\_05\_04.ppt

## Calculus Reasoning

### Outcomes

Students will be expected to

C13 Continued...

#### Achievement Indicators:

C13.8 Define and characterize the hyperbolic trigonometric functions  $\sinh x$  and  $\cosh x$  in terms of the exponential function  $e$ .

C13.9 Define  $\tanh x$ ,  $\operatorname{csch} x$ ,  $\operatorname{coth} x$ , and  $\operatorname{sech} x$  in terms of  $\sinh x$  and  $\cosh x$ .

C13.10 Determine the derivative of a hyperbolic trigonometric function.

### Elaborations – Strategies for Learning and Teaching

Students have explored both the calculus of trigonometry as well as the calculus of exponential functions. Hyperbolic functions, which provide a connection between these two areas, will now be introduced. By definition these functions are not trigonometric, but they share many similar properties and are, therefore, analogous. This course does not provide a rigorous treatment of the hyperbolic functions; rather it attempts to introduce these functions and their derivatives as a means of connection between previous topics.

Students should first be introduced to the hyperbolic sine and cosine functions as they are defined in terms of the exponential function.

Specifically:

$\sinh x = \frac{e^x - e^{-x}}{2}$	$\cosh x = \frac{e^x + e^{-x}}{2}$
------------------------------------	------------------------------------

It is not expected that students graph these functions. They should, however, realize that the domain of both  $\sinh x$  and  $\cosh x$  will be  $x \in (-\infty, \infty)$ . This is easy to demonstrate as both functions are combinations of exponentials, which are defined for all real numbers.

The other four hyperbolic trigonometric functions are most easily defined in terms of  $\sinh x$  and  $\cosh x$ . Specifically:

$\tanh x = \frac{\sinh x}{\cosh x}$	$\operatorname{csch} x = \frac{1}{\sinh x}$
$\operatorname{coth} x = \frac{\cosh x}{\sinh x}$	$\operatorname{sech} x = \frac{1}{\cosh x}$

Hyperbolic trigonometric identities are not included in this course.

Students should find the derivatives of  $\sinh x$  and  $\cosh x$  based on their definitions in terms of  $e^x$ . Specifically:

$$\begin{aligned} \frac{d}{dx} \sinh x &= \frac{d}{dx} \frac{e^x - e^{-x}}{2} \\ &= \frac{e^x - (-1)e^{-x}}{2} \\ &= \frac{e^x + e^{-x}}{2} \\ &= \cosh x \end{aligned}$$

This derivative makes the analogy of the hyperbolic functions to the standard trigonometric functions more obvious.

**General Outcome: Develop introductory calculus reasoning.**

**Suggested Assessment Strategies**

**Resources/Notes**

*Journal*

- Ask students to use the definitions of  $\sinh x$  and  $\cosh x$  to derive simplified expressions for  $\tanh x$ ,  $\coth x$ ,  $\operatorname{sech} x$ , and  $\operatorname{csch} x$  in terms of  $e^x$ .

(C13.8)

*Paper and Pencil*

- Ask students to find the exact value of each expression:

(i)  $\sinh(0)$

(ii)  $\cosh(1)$

(iii)  $\operatorname{csch}(\ln 2)$

(iv)  $\cosh(\ln 3)$

(C13.8, C13.9)

- Ask students to show that  $\tanh(2) = \frac{e^4 - 1}{e^4 + 1}$ .

(C13.9)

**Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

5.7: Hyperbolic Functions

SB: pp. 298 – 304

IG: pp. 305 - 308

PPL: SteEC\_05\_07.ppt

## Calculus Reasoning

### Outcomes

*Students will be expected to*

C13 Continued...

**Achievement Indicator:**

C13.10 *Continued*

### Elaborations – Strategies for Learning and Teaching

When differentiating  $\cosh x$ , however, students may find the results more surprising.

$$\begin{aligned}\frac{d}{dx} \cosh x &= \frac{d}{dx} \frac{e^x + e^{-x}}{2} \\ &= \frac{e^x + (-1)e^{-x}}{2} \\ &= \frac{e^x - e^{-x}}{2} \\ &= \sinh x\end{aligned}$$

Students will often guess that the derivative of  $\cosh x$  should be  $-\sinh x$ . It is important that they determine these derivatives to see why this is not the case.

Based on the definitions of the other four hyperbolic trigonometric functions, their derivatives can be developed using basic differentiation rules. A summary of the derivatives of the hyperbolic trigonometric functions is given here:

$\frac{d}{dx} \sinh x = \cosh x$	$\frac{d}{dx} \cosh x = \sinh x$	$\frac{d}{dx} \tanh x = \operatorname{sech}^2 x$
$\frac{d}{dx} \operatorname{csch} x = -\operatorname{csch} x \coth x$	$\frac{d}{dx} \operatorname{sech} x = -\operatorname{sech} x \tanh x$	$\frac{d}{dx} \coth x = -\operatorname{csch}^2 x$

Inverse hyperbolic trigonometric functions are not included in this course. Students should, however, be exposed to differentiating the hyperbolic trigonometric functions in combination with other known functions.



**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Paper and Pencil*

- Ask students to differentiate the following

(i)  $y = \sinh(\cosh x)$

(ii)  $y = \tanh(\ln x)$

(iii)  $y = \coth(x)\operatorname{sech}(x)$

(iv)  $y = \sqrt{\cosh(x-5)}$

(v)  $y = \frac{\operatorname{csch}(x)}{\cos(x)}$

(vi)  $y = \sin(\sinh(x))$

(C13.10)

**Resources/Notes****Authorized Resource***Single Variable Essential Calculus  
Second Edition*

5.7: Hyperbolic Functions

SB: pp. 298 – 304

IG: pp. 305 - 308

PPL: SteEC\_05\_07.ppt



Antidifferentiation and Integration

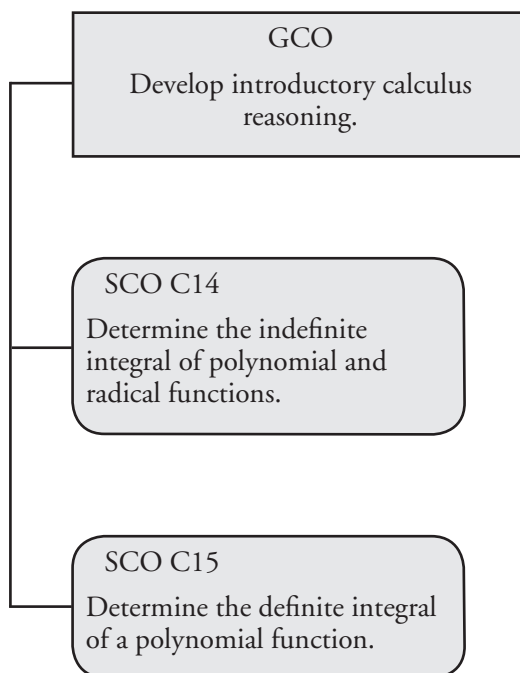
Suggested Time: 12 Hours

# Unit Overview

## Focus and Context

In this unit, students will make the connection between differentiation and antidifferentiation using the Fundamental Theorem of Calculus. When antidifferentiating, the Power Rule is often used. Students will estimate area under a curve on interval  $[a, b]$  using subintervals. Riemann Sums will then be used to determine the exact area. Students will apply the Fundamental Theorem of Calculus to determine the net area under a curve and the area between two curves.

## Outcomes Framework



## SCO Continuum

Mathematics 2200	Mathematics 3200	Calculus 3208
Relations and Functions	Relations and Functions	Calculus
not addressed	not addressed	<p>C14 Determine the indefinite integral of polynomial and radical functions. [C, CN, PS, R]</p> <p>C15 Determine the definite integral of a polynomial function. [C, CN, PS, R]</p>

## Mathematical Processes

- |  |                      |
|--|----------------------|
| [C] Communication                      | [PS] Problem Solving |
| [CN] Connections                       | [R] Reasoning        |
| [ME] Mental Mathematics and Estimation | [T] Technology       |
|  | [V] Visualization    |

## Calculus Reasoning

### Outcomes

*Students will be expected to*

**C14 Determine the indefinite integral of polynomial and radical functions.**

[C, CN, PS, R]

### Achievement Indicators:

C14.1 *Explain the meaning of the phrase “ $F(x)$  is an antiderivative of  $f(x)$ ”.*

C14.2 *Determine the general antiderivative of functions.*

### Elaborations—Strategies for Learning and Teaching

Taking the indefinite integral is simply reversing differentiation in much the same way that division reverses multiplication. Students will be exposed to the notation, concepts and properties of the indefinite integral, limited to polynomial and radical functions.

So far in this course, students have been given a function and asked to find its derivative. The process is now reversed; given the derivative, students will determine the original function.

Ask students to think about a differentiable function  $F(x)$  where  $\frac{d}{dx} F(x) = 2x$ . Ask them to identify a function they would have to differentiate to get this result. They should notice that there are a whole set of functions such as:

$$\frac{d}{dx}(x^2) = 2x$$

$$\frac{d}{dx}(x^2 + 2) = 2x$$

$$\frac{d}{dx}(x^2 + 8) = 2x$$

$$\frac{d}{dx}(x^2 - 11) = 2x$$

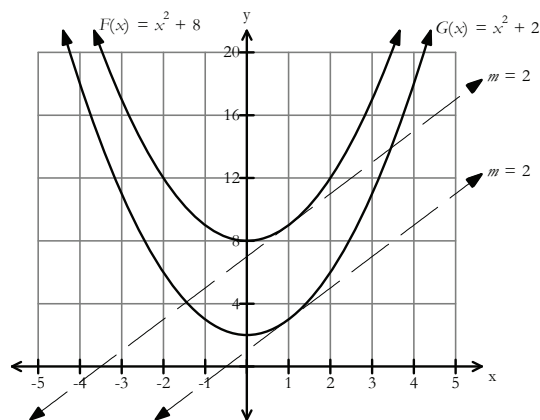
Discuss with students that since the derivative of a constant is zero, an infinite number of antiderivatives exist. The antiderivative of  $2x$  equals  $x^2 + C$ , where  $C$  is a constant.

Students should be able to distinguish between differentiation and antidifferentiation. Differentiating a function results in a single function whereas antidifferentiating results in a family of functions—all of which have the same derivative.

Introduce students to the proper notation when finding antiderivatives. The antiderivative of a function  $f(x)$  is a differentiable function  $F(x)$  whose derivative is equal to  $f(x)$  on a given interval (i.e.,  $F'(x) = f(x)$ ).

**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Observation*

- Using graphing technology, ask students to graph  $F(x) = x^2 + 8$  and  $G(x) = x^2 + 2$  on the same axis and to construct the tangent lines at  $x = 1$ . Ask them the following questions:
  - What is the equation of the tangent line to the curve  $F(x)$  at  $x = 1$ ? What is the equation of the tangent line to the curve  $G(x)$  at  $x = 1$ ?
  - Why is the  $\frac{d}{dx}(x^2 + 2) = \frac{d}{dx}(x^2 + 8)$ . What is the simplified derivative?
  - Why do functions, which vary by a constant, have the same slope (e.g.,  $m = 2$ ) for tangents drawn at the same value of  $x$ .



Ask students why the graphs of antiderivatives of a given function are vertical translations of each other (i.e., family of curves).

(C14.1, C14.2)

**Resources/Notes****Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**3.7: Antiderivatives**

Student Book (SB): pp. 189 - 198  
Instructor's Guide (IG): pp. 199  
- 215

Power Point Lectures (PPL):  
SteEC\_03\_07.ppt

## Calculus Reasoning

### Outcomes

Students will be expected to

C14 Continued...

#### Achievement Indicators:

C14.3 Use antiderivative notation appropriately (i.e.,  $\int f(x) dx$  for the general antiderivative of  $f(x)$ ).

C14.4 Identify the properties of antiderivation.

C14.2 Continued

### Elaborations—Strategies for Learning and Teaching

The process of finding  $F(x)$  given  $f(x)$ , is called antidifferentiation (i.e., undoing the derivative) or integration. Introduce students to the notation of the indefinite integral  $\int f(x) dx = F(x) + C$ . This is equivalent to  $\frac{d}{dx}[F(x) + C] = f(x)$  where  $f(x)$  is called the integrand and  $dx$  indicates the variable of integration. When students are asked, for example, to find all the functions whose derivative is  $2x$ , this can now be expressed as  $\int 2x dx$ . The symbol  $\int$  represents determining the general antiderivative.

Before students are formally introduced to the rules for antidifferentiation, provide them with an opportunity to make conjectures about a variety of functions. Ask students to discuss possible antiderivatives for the following functions and to verify their results using differentiation.

- (i)  $f(x) = 5$
- (ii)  $g(x) = 4x$
- (iii)  $h(x) = 3x^2$
- (iv)  $k(x) = 3x^2 + 4x + 5$

Students should make the following connections:

- The antiderivative of  $g(x) = 4x$ , for example, is the same as 4 multiplied by the antiderivative of  $x$  (i.e.,  $\int 4x dx = 4 \int x dx$ ).
- The antiderivative of a sum is the sum of the antiderivatives (i.e.,  $\int (3x^2 + 4x + 5) dx = \int 3x^2 dx + \int 4x dx + \int 5 dx$ ).

Students worked with the constant property and the sum and difference property when they were using limits and derivatives. They will continue to apply these properties with integral notation.

$$\int c \cdot f(x) dx = c \cdot \int f(x) dx$$

$$\int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx$$

Ask students to continue to identify patterns as they determine the antiderivative of functions such as  $x^2$ ,  $x^3$  and  $x^4$ . This process should lead into a discussion around the development of the Power Rule for antiderivatives. When asked to find the antiderivative of a variable raised to a power ( $x^n$ ), students must add one to the exponent and divide by this new exponent (i.e.,  $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ ).



**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Journal*

- Davin stated that the solution for  $\frac{d}{dx} \int (x^2 + 4x - 7) dx$  is the same as  $\int \left[ \frac{d}{dx} (x^2 + 4x - 7) \right] dx$ .

Ask students if they agree or disagree with this statement. They should justify their answer.

(C14.2)

*Paper and Pencil*

- Ask students to determine the general antiderivative for the following:

(i)  $f'(x) = 7x + 13$

(ii)  $g'(x) = 3x^{12} + 7x^8 - 11$

(iii)  $k'(x) = \sqrt{x}$

(iv)  $y' = \frac{3}{x^2}$

(C14.2, C14.3, C14.4)

- Ask students to integrate the following:

(i)  $\int (x-1)^2 dx$

(ii)  $\int \frac{x^2 - 9}{x + 3} dx$

(iii)  $\int \frac{3x^{\frac{1}{3}} - 2x^{\frac{1}{2}}}{x^{\frac{1}{6}}} dx$

(C14.2, C14.3, C14.4)

**Resources/Notes****Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**3.7: Antiderivatives**

SB: pp. 189 - 198

IG: pp. 199 - 215

PPL: SteEC\_03\_07.ppt

## Calculus Reasoning

### Outcomes

*Students will be expected to*

C14 Continued...

### Achievement Indicators:

C14.2, C14.4 *Continued*

### Elaborations – Strategies for Learning and Teaching

Ask students to find the antiderivative of  $x^{-1}$ . They should recognize division by zero is undefined. The Power Rule does not work, therefore, when  $n = -1$ .

The following rules should be used to compute antiderivatives of the sum and difference of functions. Limit examples to polynomial functions, and functions containing terms involving  $x^{\frac{m}{n}}$ , where  $m$  and  $n$  are integers,  $n \neq 0$ .

$$\int k dx = kx + C$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \text{ if } n \neq -1$$

Ask students to find  $F(x)$  for functions such as the following using the Constant Rule and Power Rule:

(i)  $f(x) = 3x^{12} + 7x - 11$

(ii)  $f(x) = x\sqrt{x}$

(iii)  $f(x) = \frac{4}{\sqrt[3]{x^5}}$

(iv)  $f(x) = (2x - 1)^2$

Students are only responsible for integrals where expanding or simplifying expressions produces terms which can be integrated by the power rule. Integration by substitution is not an outcome of this course.

**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Paper and Pencil*

- Ask students to work in groups to play the following puzzle game involving derivatives and antiderivatives. The objective is to place each puzzle piece in a 3 by 3 grid such that the derivative on one puzzle piece lies adjacent to its antiderivative on another puzzle piece. Provide each group with all nine puzzle pieces.

$$\begin{array}{ccc} x^2 + 5x + 1 & & \\ 8x & \frac{2}{3}x^3 - \frac{5}{2}x^2 & \\ \frac{1}{2}x^2 - 5x + 2 & & \end{array}$$

$$\begin{array}{ccc} \frac{1}{x^3} & & \\ \frac{-21}{x^4} + \frac{8}{x^3} & \frac{84}{x^3} - \frac{24}{x^2} & \\ x^5 - x^3 + 4 & & \end{array}$$

$$\begin{array}{ccc} \frac{-5}{x^2} - 3\sqrt{5} & & \\ 8x + 1 & & -5 \\ 3x^2 + 1 & & \end{array}$$

$$\begin{array}{ccc} \frac{1}{4}x^4 + \frac{1}{2}x^2 + \frac{1}{8} & & \\ 2x^2 - 5x & 3\sqrt{5} - 5\sqrt{x} & \\ \frac{5}{x} - 3\sqrt{5}x & & \end{array}$$

$$\begin{array}{ccc} \frac{1}{6}x^3 - \frac{5}{2}x^2 + 2x + 7 & & \\ -5x & & 8 \\ 2 & & \end{array}$$

$$\begin{array}{ccc} 4x^2 - x & & \\ -5x + 40 & & 8 \\ 2x - 5 & & \end{array}$$

$$\begin{array}{ccc} \frac{5}{x^2} & & \\ \frac{2}{3}\sqrt{x} & & x^5 - x \\ 2x + 5 & & \end{array}$$

$$\begin{array}{ccc} \frac{1}{3} & & \\ 5x^4 - 1 & \frac{7}{x^3} - \frac{4}{x^2} & \\ x^3 + x & & \end{array}$$

$$\begin{array}{ccc} 5x^4 - 3x^2 & & \\ \frac{-5\sqrt{x}}{2x} & & 3x^2 + x \\ 8x - 1 & & \end{array}$$

(C14.2, C14.4)

**Resources/Notes****Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**3.7: Antiderivatives**

SB: pp. 189 - 198

IG: pp. 199 - 215

PPL: SteEC\_03\_07.ppt

**Suggested Resource**

Resource Link:

[www.k12pl.nl.ca/curr/10-12/math/3208/res-link/anti-int.html](http://www.k12pl.nl.ca/curr/10-12/math/3208/res-link/anti-int.html)

- print ready derivative puzzle

## Calculus Reasoning

### Outcomes

*Students will be expected to*

C14 Continued...

#### Achievement Indicators:

C14.5 *Determine the indefinite integral of a function given extra conditions.*

C14.6 *Use antidifferentiation to solve problems about motion of a particle along a line that involves:*

- *computing the position given the initial position and velocity as a function of time*
- *computing velocity and/or position given the suitable initial conditions and acceleration as a function of time.*

### Elaborations—Strategies for Learning and Teaching

Using the rules of integration, students will be provided with specific information allowing them to determine the value of the constant  $C$ . If they are given initial conditions, they can find specific antiderivatives rather than general ones.

Ask students to work through examples such as the following:

- Find  $f(x)$  if  $f'(x) = 2x + 1$  where  $f(0) = 3$ .
- Find  $f(x)$  if  $f''(x) = 12x^2 - 8$  and the slope of the tangent line at the point  $(2, 3)$  is  $-8$ . Ensure students recognize that there are two conditions given:  $f(2) = 3$  and  $f'(2) = -8$ .

Discuss with students that when they determine the value of  $C$ , they specify one member of the family of functions which satisfies the extra conditions (i.e., initial conditions).

There are many “initial value” application problems that could be covered here. Biologists may want to determine the future population given the current rate of population growth. A banker may want to determine future value of a mortgage given current interest rates. For the purpose of this course, the problems should focus exclusively on motion problems.

In the Derivatives unit, students were exposed to position ( $s$ ), velocity ( $v$ ) and acceleration ( $a$ ) with respect to time ( $t$ ) where  $v(t) = s'(t)$  and  $a(t) = v'(t) = s''(t)$ . They should now apply integration to determine the position function given the velocity function and extra conditions. Similarly, they will determine the velocity and position function given the acceleration function with extra conditions.

Ask students to solve a problem, such as the one shown. The following questions could be used to guide students:

- The velocity of a ball being thrown in the air from an initial height of 1 m is given by  $v(t) = -9.8t + 12$ , where  $t$  is the time in seconds. Determine the quadratic function that models the height of the ball after  $t$  seconds. Use this function to calculate the height of the ball after 2.5 seconds.

(i) What will the antiderivative of  $v(t)$  produce?

$$h(t) = \int (-9.8t + 12) dt$$

$$h(t) = -4.9t^2 + 12t + C$$

**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Paper and Pencil*

- Ask students to answer the following:
  - (i) Find  $f(x)$  if  $f''(x) = 6x - 4$ ,  $f'(1) = 4$  and  $f(2) = 9$ .
  - (ii) Find  $y$  if  $y' = \frac{1}{\sqrt{x}}$  and  $(4, -1)$  is a point on the graph of  $y$ .
  - (iii) Find  $h(t)$  if  $h'(t) = -10t + 20$  and  $h(2) = 120$ .
  - (iv) If  $g'(x) = \frac{1}{\sqrt{x^3}} + x^{-2} - x^7$  and  $g(1) = -\frac{1}{8}$ , find the equation for  $g(x)$ .

(C14.5)

- A particle moves along a straight line with an acceleration of  $a(t) = t^2 + 2$ . If at  $t = 0$  the particle is stationary, ask students to write an equation describing its velocity,  $v(t)$ .

(C14.6)

**Resources/Notes****Authorized Resource***Single Variable Essential Calculus  
Second Edition***3.7: Antiderivatives**

SB: pp. 189 - 198

IG: pp. 199 - 215

PPL: SteEC\_03\_07.ppt

Video Examples: 4 and 5

## Calculus Reasoning

### Outcomes

*Students will be expected to*

C14 Continued...

### Achievement Indicator:

C14.6 *Continued*

### Elaborations – Strategies for Learning and Teaching

- (ii) What is the initial condition? How can this be used to find the constant C?

$$1 = -4.9(0)^2 + 12(0) + C$$

$$C = 1$$

- (iii) What is the resulting equation for  $h(t)$ ? What is the height after 2.5 seconds?

$$h(t) = -4.9t^2 + 12t + 1$$

$$h(2.5) = -4.9(2.5)^2 + 12(2.5) + 1$$

$$h(2.5) = 0.375 \text{ m}$$

Students should also solve problems, such as the following, where they have to determine the position function given the acceleration function:

- A particle is accelerated along a line so that its acceleration, in  $\frac{\text{m}}{\text{s}^2}$ , is  $a(t) = \frac{1}{2\sqrt{t}}$ . Determine the position of the particle at time,  $t$ , if  $s(0) = v(0) = 0$ .
- A particle moves along a line and has acceleration given by  $a(t) = 6t + 4$ . If its initial velocity is -6 centimetres per second and its initial position is 9 centimetres, find the position function.

## General Outcome: Develop introductory calculus reasoning.

### Suggested Assessment Strategies

#### *Paper and Pencil*

- Ask students to answer the following:
  - (i) If the acceleration of a particle moving along the  $x$ -axis is given by  $a(t) = 12t$ , determine its position at  $t = 2$ , if  $v(t) = 22$  and  $x(0) = 0$ .  
(C14.6)
  
  - (ii) The height of a ball thrown straight up with an initial speed of  $80 \frac{\text{ft}}{\text{sec}}$  from a rooftop 96 feet high is  $s(t) = -16t^2 + 80t + 96$  where  $t$  is the elapsed time that the ball is in the air. The ball misses the rooftop on its way down and eventually strikes the ground.
    - (a) When does the ball strike the ground?
    - (b) At what time  $t$  will the ball pass the rooftop on its way down?
    - (c) What is the velocity of the ball at  $t = 2$ ?
    - (d) When is the velocity of the ball equal to zero?
    - (e) What is the velocity of the ball as it passes the rooftop on the way down?
    - (f) What is the velocity of the ball when it strikes the ground?  
(C14.6)
  
  - (iii) A ball is thrown straight up from a height of 20 m with an initial velocity of  $40 \frac{\text{m}}{\text{s}}$  and acceleration is  $-10 \frac{\text{m}}{\text{s}^2}$ .
    - (a) At what time will the ball reach its maximum height?
    - (b) At what time will it hit the ground?
    - (c) What is the velocity of the ball when it strikes the ground?
    - (d) What is the total distance travelled by the ball?  
(C14.6)

### Resources/Notes

#### Authorized Resource

*Single Variable Essential Calculus  
Second Edition*

#### 3.7: Antiderivatives

SB: pp. 189 - 198

IG: pp. 199 - 215

PPL: SteEC\_03\_07.ppt

Video Examples: 4 and 5

## Calculus Reasoning

### Outcomes

*Students will be expected to*

**C15 Determine the definite integral of a polynomial function.**

[C, CN, PS, R]

### Achievement Indicators:

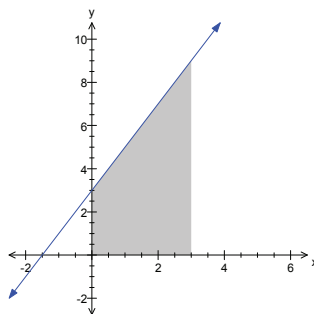
C15.1 *Estimate an area using a finite sum.*

C15.2 *Determine the area using the infinite Riemann sum.*

### Elaborations—Strategies for Learning and Teaching

For this outcome, students will apply the Fundamental Theorem of Calculus to determine the area under a curve. While this topic can be very challenging, the intent of this course is to expose students to the topic at an introductory level.

Students should be familiar with the idea of finding the area of composite shapes that are made up of "straight" lines. This would be a good place to start discussion. Ask students to determine the area, for example, of the shaded region below the line  $y = 2x + 3$  and above the  $x$ -axis from  $x = 0$  to  $x = 3$ .

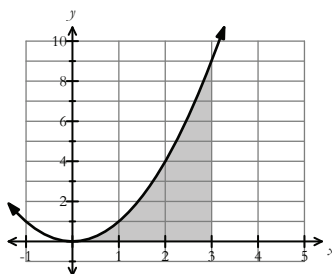


Students may notice that the total area can be viewed as the area of a trapezoid or as a sum of the areas of a rectangle and a triangle. The challenge arises when the function is something other than linear.

Introduce students to a method for approximating areas under curves. When working with a continuous function that is non-negative on an interval  $[a, b]$ , the goal is for students to approximate the area of the region bounded by the graph of  $f(x)$  and the  $x$ -axis from  $x = a$  to  $x = b$ .

Provide students with an example such as the following:

- Determine the area under the curve  $y = x^2$  and above the  $x$ -axis from  $x = 0$  to  $x = 3$ .



One strategy is for students to divide the interval into subintervals and approximate the area using rectangles built to touch the function on the subintervals. The area of each rectangle is calculated and summed. Discuss with students that this strategy only produces an approximation to the area. This approximation, however, improves as the number of subintervals increase.

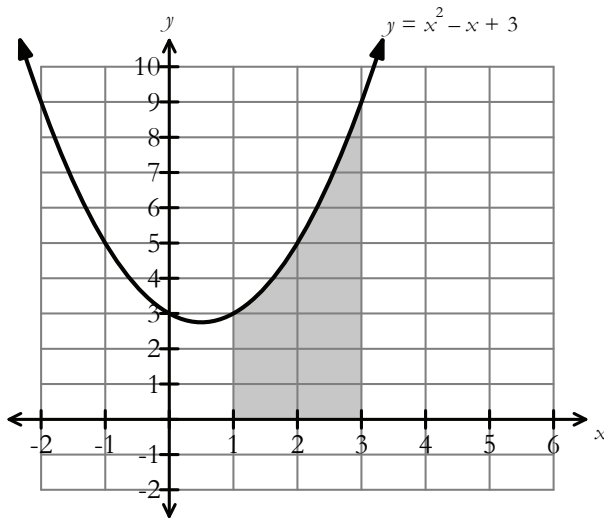


**General Outcome: Develop introductory calculus reasoning.**

### Suggested Assessment Strategies

#### *Paper and Pencil*

- Ask students to estimate the area under the curve  $y = x^2 - x + 3$  where  $1 \leq x \leq 3$  using 6 right endpoint rectangles.



Ask students to consider under which conditions the right endpoint method overestimates the actual area and under which conditions it underestimates the actual area.

(C15.1, C15.2)

### Resources/Notes

#### Authorized Resource

*Single Variable Essential Calculus  
Second Edition*

4.1: Areas and Distances

SB: pp. 199 - 210

IG: pp. 217 - 224

PPL: SteEC\_04\_01.ppt

Video Examples: 1

**Calculus Reasoning**

**Outcomes**

*Students will be expected to*

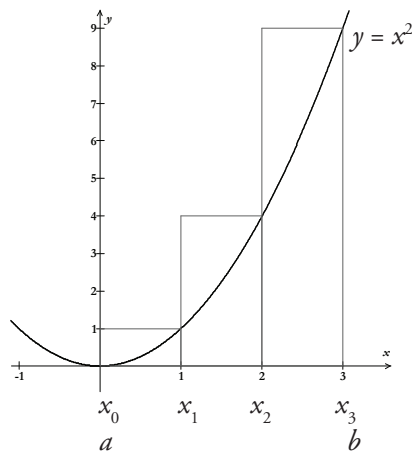
C15 Continued...

**Achievement Indicators:**

C15.1, C15.2 *Continued*

**Elaborations – Strategies for Learning and Teaching**

To simplify the explanation and the calculations, the interval  $[a, b]$  should be divided into subintervals of equal width, where the sample points correspond to the right endpoints of the subintervals. Begin by asking students to divide  $[0, 3]$  into  $n = 3$  subintervals  $[0, 1]$ ,  $[1, 2]$  and  $[2, 3]$ . Students should recognize the width of each subinterval, denoted as  $\Delta x$ , is 1 unit and can be calculated by dividing the width of the interval by  $n$  subintervals (i.e.,  $\Delta x = \frac{b-a}{n}$ ).



Students should also recognize that the height is given by the function value at a specific point in the interval (i.e.,  $1^2$ ,  $2^2$  and  $3^2$ ). Ask students why the value  $x_0 = 0$  was not used in their calculation. By determining the area of each rectangle and summing the areas of the rectangles, students obtain an approximation to the area, called a Riemann sum. Ask students to let  $S_3$  represent the sum of the areas of the three rectangles. The sigma notation is new to students and should be briefly explained. A more in-depth treatment of the sigma notation may be found in Appendix B.

$$S_3 = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x$$

This can be more compactly written as  $S_3 = \sum_{i=1}^3 f(x_i)\Delta x$  where  $i$  counts from 1 to 3 and  $\Sigma$  means to sum.

Using right endpoints of the subintervals would produce

$$S_3 = (1 \cdot 1) + (4 \cdot 1) + (9 \cdot 1) = 14$$

Using left endpoints of the subintervals would produce

$$S_3 = (0 \cdot 1) + (1 \cdot 1) + (4 \cdot 1) = 5$$

Using midpoints of the subintervals would produce

$$S_3 = \left(\frac{1}{4} \cdot 1\right) + \left(\frac{9}{4} \cdot 1\right) + \left(\frac{25}{4} \cdot 1\right) = \frac{35}{4}$$

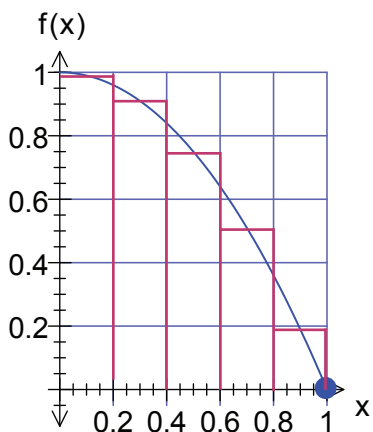
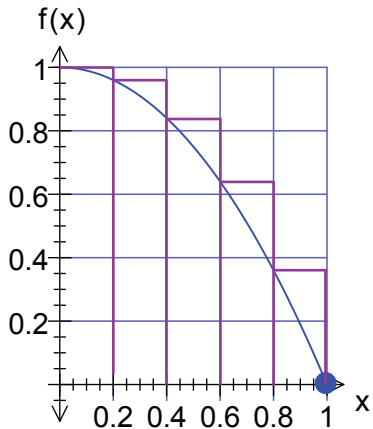
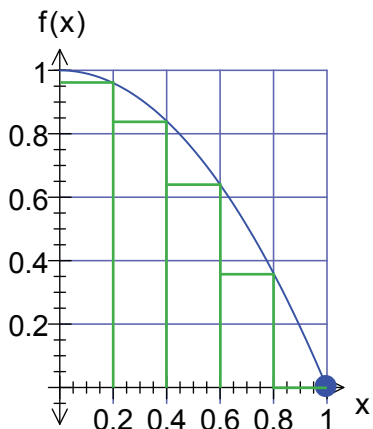
In general, a Riemann sum with  $n$  rectangles with  $\Delta x = \frac{b-a}{n}$  can be determined by  $S_n = \sum_{i=1}^n f(x_i)\Delta x$ .

**General Outcome: Develop introductory calculus reasoning.**

**Suggested Assessment Strategies**

*Paper and Pencil*

- Ask students to estimate the area under the curve  $y = 1 - x^2$  where  $0 \leq x \leq 1$  using five right endpoint rectangles, five left endpoint rectangles, and five midpoint rectangles.



**Resources/Notes**

**Authorized Resource**

*Single Variable Essential Calculus Second Edition*

**4.1: Areas and Distances**

SB: pp. 199 - 210

IG: pp. 217 - 224

PPL: SteEC\_04\_01.ppt

Tools for Enriching Calculus Animations (TEC): Visual 4.1: Approximating the Area under a Parabola

Video Examples: 1

**Note:**

More in-depth exploration of Riemann sums is available in Appendix B, pp. 247-250.

(C15.1, C15.2)

**Calculus Reasoning**

**Outcomes**

*Students will be expected to*

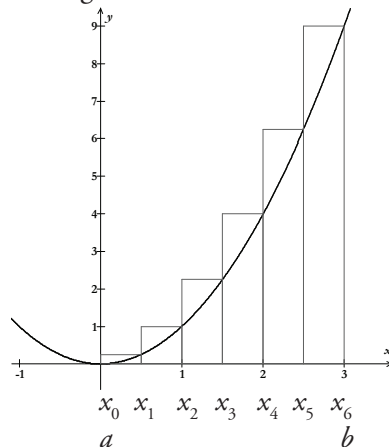
C15 Continued...

**Achievement Indicators:**

C15.1, C15.2 *Continued*

**Elaborations – Strategies for Learning and Teaching**

Ask students to further investigate the area using 6 right endpoint rectangles.



The following questions could be used to facilitate discussion:

- What is the total width of the interval? If there are six rectangles, what is the width of each rectangle?
- Which values should be used to determine the height of each rectangle?
- What is the area of each rectangle?
- What is the sum of all six rectangles ( $S_6$ )?

$$S_6 = \sum_{i=1}^6 f(x_i)\Delta x$$

$$S_6 = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x$$

$$S_6 = (\frac{1}{4} \cdot \frac{1}{2}) + (1 \cdot \frac{1}{2}) + (\frac{9}{4} \cdot \frac{1}{2}) + (4 \cdot \frac{1}{2}) + (\frac{25}{4} \cdot \frac{1}{2}) + (9 \cdot \frac{1}{2}) = 11.375$$

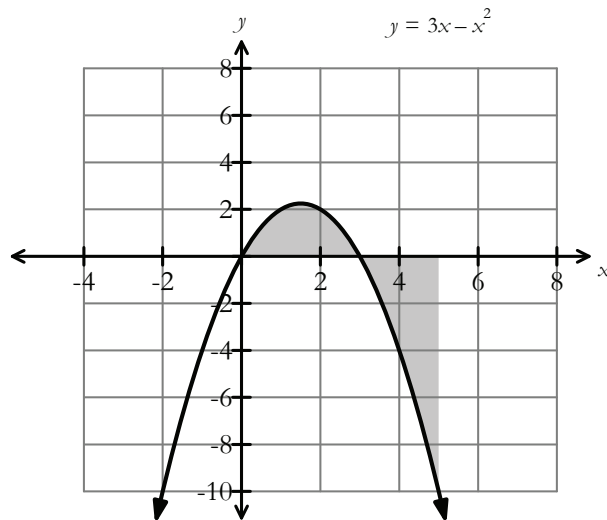
Ask students to repeat this investigation using large number of sub-intervals. With the aid of technology, they should generate the area approximations using sub-intervals of 12, 24, 48, 100 and 1000. The chart below illustrates that, as the number of sub-intervals increase, there is an increase in the accuracy of the estimation.

Number of sub-intervals ( $n$ )	Sum of the areas of the rectangles
3	14
6	11.375
12	10.15625
24	9.5703125
48	9.283203125
100	9.13545
1000	9.0135045

Discuss with students that, as  $n$  increases, the sums converge to a limit, which is the whole idea of integral calculus.

**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Paper and Pencil*

- Ask students to answer the following:
  - Determine the value of a Riemann sum for  $y = 3x - x^2$  from  $0 \leq x \leq 5$  with  $\Delta x = 1$ .



- Why is the value of the Riemann sum negative?
- Use the Riemann sum on  $[0,3]$  and  $[3,5]$  to determine the actual area between  $y = 3x - x^2$  and the  $x$ -axis.

(C15.1, C15.2)

**Resources/Notes****Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**4.1: Areas and Distances**

SB: pp. 199 - 210

IG: pp. 217 - 224

PPL: SteEC\_04\_01.ppt

Video Examples: 1

**Suggested Resource**

Resource Link:

[www.k12pl.nl.ca/curr/10-12/math/3208/res-link/anti-int.html](http://www.k12pl.nl.ca/curr/10-12/math/3208/res-link/anti-int.html)

- interactive Riemann sums generator

## Calculus Reasoning

### Outcomes

*Students will be expected to*

C15 Continued...

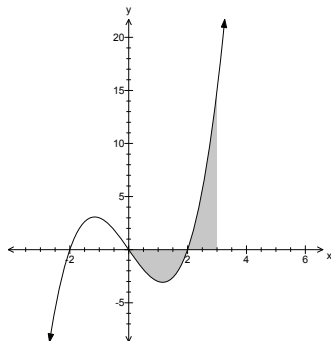
#### Achievement Indicators:

C15.1, C15.2 *Continued*

C15.3 *Convert a Riemann sum to a definite integral.*

### Elaborations—Strategies for Learning and Teaching

It is important for students to explore situations where the function takes on both positive and negative values.



Since a Riemann sum is the sum of the products of widths and heights of the rectangles with dimensions  $f(x)$  by  $\Delta x$ , the product will be positive where  $f(x) > 0$  and negative where  $f(x) < 0$ . The combined net accumulation can be positive, negative or zero.

The Riemann sum is one strategy to approximate the accumulation of widths times heights of a region bounded by a curve  $y = f(x)$  and the  $x$ -axis on the interval  $[a, b]$ . As larger values of subintervals  $n$  are chosen and the width of each rectangle decreases, the approximation becomes more accurate. Taking the limit at infinity of the Riemann sum, students are able to calculate the exact area. Introduce this limit to students as the definite integral.

If  $f(x)$  is a function defined on  $[a, b]$ , the definite integral of  $f(x)$  from  $a$  to  $b$  is the number

$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i)\Delta x$$

provided the limit exists. If it does exist,  $f(x)$  is integrable on  $[a, b]$ .

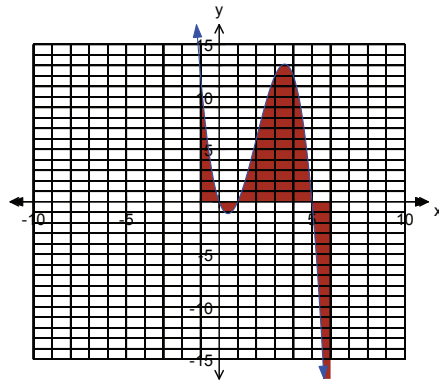
Provide students with examples where they identify the function and express the limit as a definite integral. They should discuss what the definite integral represents geometrically.

**General Outcome: Develop introductory calculus reasoning.**

### Suggested Assessment Strategies

*Paper and Pencil*

- Ask students to identify which areas are positive and which are negative.



(C15.1, C15.2)

### Resources/Notes

#### Authorized Resource

*Single Variable Essential Calculus  
Second Edition*

#### 4.2: The Definite Integral

SB: pp. 210 - 223

IG: pp. 225 - 237

PPL: SteEC\_04\_02.ppt

TEC: Module 4.2/M6.5

Estimating Areas under Curves

Visual 4.2: Approximating an  
Integral with Riemann Sums

#### Note:

More in-depth exploration of Riemann sums is available in Appendix B, pp. 247-250.

## Calculus Reasoning

### Outcomes

Students will be expected to

C15 Continued...

#### Achievement Indicators:

C15.4 Using definite integrals, determine the area under a polynomial function from  $x = a$  to  $x = b$ .

C15.5 Calculate the definite integral of a function over an interval  $[a, b]$ .

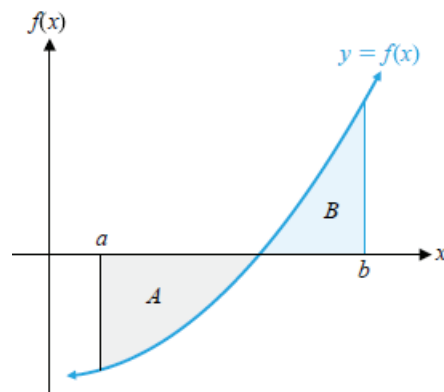
### Elaborations – Strategies for Learning and Teaching

Evaluating definite integrals using limits of Riemann sums can be challenging especially if the calculations involve polynomials of high degree. A more efficient method to evaluate definite integrals involves the Fundamental Theorem of Calculus.

- If  $f(x)$  is continuous on  $[a, b]$  and  $F(x)$  is any antiderivative of  $f(x)$  then

$$\int_a^b f(x) dx = F(b) - F(a)$$

This represents the net accumulated area between the graph of  $f(x)$  and the  $x$ -axis from  $x = a$  to  $x = b$  where the areas above the  $x$ -axis are counted positively and the areas below the  $x$ -axis are counted negatively.



$$\int_a^b f(x) dx = -A + B$$

To determine the actual area students should use  $B - A$ .

Ensure students recognize the difference between an indefinite integral and a definite integral. The indefinite integral is a function or a family of functions. The definite integral is a number.



**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Paper and Pencil*

- Ask students to answer the following:
  - (i) Calculate area under the curve for the indicated intervals:
    - (a)  $f(x) = 2x^3 + 1$ ,  $[0, 2]$
    - (b)  $f(x) = 16 - x^4$ ,  $[-1, 0]$
    - (c)  $f(x) = x^3 - 6x^2 + 8x + 6$ ,  $[0, 4]$
  - (ii) Ask students to calculate the net accumulation between the graph of  $f(x) = 4 - x^2$  and the  $x$  - axis in the interval  $[-4, 4]$ . They should explain their result using a graph. Ask students to find the total area and to examine how symmetry could be used in this problem.

(C15.4, C15.5)

**Resources/Notes****Authorized Resource***Single Variable Essential Calculus  
Second Edition***4.3: Evaluating Definite Integrals**

SB: pp. 223 - 232

IG: pp. 238 - 244

PPL: SteEC\_04\_03.ppt

Video Examples: 1

## Calculus Reasoning

### Outcomes

Students will be expected to

C15 Continued...

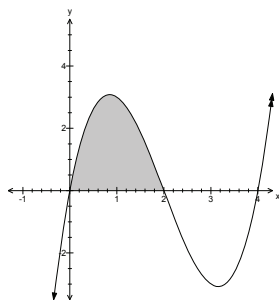
### Achievement Indicators:

C15.4, C15.5 Continued

### Elaborations—Strategies for Learning and Teaching

Students should distinguish between finding the area under a curve  $f(x)$  bounded by the  $x$ -axis when  $f(x)$  is above the  $x$ -axis, below the  $x$ -axis or a combination of both.

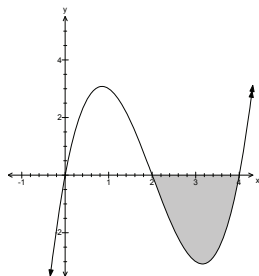
- Consider when  $f(x) \geq 0$



$$\int_0^2 (x^3 - 6x^2 + 8x) dx = \left. \frac{x^4}{4} - 2x^3 + 4x^2 \right|_0^2 = 4$$

This represents an accumulation of 4 units<sup>2</sup> above the  $x$ -axis.

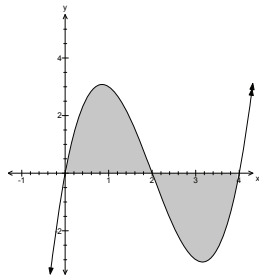
- Consider  $f(x) \leq 0$



$$\int_2^4 (x^3 - 6x^2 + 8x) dx = \left. \frac{x^4}{4} - 2x^3 + 4x^2 \right|_2^4 = -4$$

This represents an accumulation of 4 units<sup>2</sup> under the  $x$ -axis

- Consider if  $f$  is on the interval  $[0, 4]$



$$\int_0^4 (x^3 - 6x^2 + 8x) dx = \left. \frac{x^4}{4} - 2x^3 + 4x^2 \right|_0^4 = 0$$

This represents a net accumulation of 0.

This would be a good opportunity to compare net accumulation and total area. Net accumulation can be determined by evaluating the definite integral. If asked to find the total area, students should add positive accumulations with the opposite of the negative accumulations.

**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Paper and Pencil*

- (i) The function  $f(x)$  is continuous on the closed interval  $[a, b]$ . The following information is given with  $a < c < d < b$ .

$$[a, c] \quad f(x) \leq 0 \quad \int_a^c f(x) dx = -5$$

$$[c, d] \quad f(x) \leq 0 \quad \int_c^d f(x) dx = -2$$

$$[d, b] \quad f(x) \geq 0 \quad \int_d^b f(x) dx = 3$$

Students were asked to determine the net accumulation between  $f(x)$  and the  $x$ -axis on the interval  $[a, b]$ . Five students provided the following answers:

Mary: 10

Jane: 0

Lisa: - 4

Jill: - 10

Abby: 6

Ask students who they agree with and why.

(C15.4, C15.5)

- (iii) Ask students to determine the value of  $c$  :

$$\int_{-3}^0 (x^4 + 2x + c) dx = 200$$

(C15.4, C15.5)

**Resources/Notes****Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**4.3: Evaluating Definite Integrals**

SB: pp. 223 - 232

IG: pp. 238 - 244

PPL: SteEC\_04\_03.ppt

Video Examples: 1

## Calculus Reasoning

### Outcomes

Students will be expected to

C15 Continued...

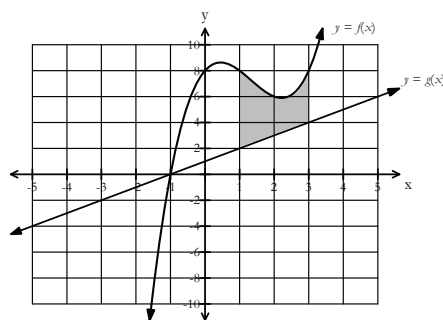
#### Achievement Indicator:

C15.6 Determine the area between two polynomial functions.

### Elaborations—Strategies for Learning and Teaching

Students have calculated the area of regions that lie under the graphs of functions. They will now use integrals to find areas of regions that lie between the graphs of two polynomial functions.

Using functions such as  $f(x) = x^3 - 4x^2 + 3x + 8$  and  $g(x) = x + 1$ , ask students to determine the area between the two functions on  $[1, 3]$ . Encourage them to sketch the functions.



Discuss with students that  $f(x)$  and  $g(x)$  are continuous functions with the property that for all  $x$  in  $[1, 3]$ ,  $f(x) \geq g(x)$ . They should recognize that the area between the two functions, bounded by two vertical lines  $x = 1$  and  $x = 3$  can be found by subtracting the area under the graph of  $g(x)$  from the area under the graph of  $f(x)$ . The problem of finding the area between two curves  $f(x)$  and  $g(x)$  between  $x = a$  and  $x = b$  can be reduced to one of finding the area under the non-negative function  $h(x) = f(x) - g(x)$  between  $x = a$  and  $x = b$ .

$$\begin{aligned} \text{Area} &= \int_a^b [f(x) - g(x)] dx \\ &= \int_1^3 [(x^3 - 4x^2 + 3x + 8) - (x + 1)] dx \\ &= \frac{22}{3} \end{aligned}$$

One of the more common mistakes students make with these problems is to neglect parentheses around the second function.

There are situations where the area is bounded by two curves intersecting more than twice, resulting in two or more distinct regions. For each region, students should determine which is the upper curve.

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**General Outcome: Develop introductory calculus reasoning.**

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**Suggested Assessment Strategies**

*Paper and Pencil*

- Ask students to determine the area bounded by:
  - (i)  $y = x^2 + 2$ ,  $y = -x$ ,  $x = 0$  and  $x = 2$ .
  - (ii)  $y = -x^3 + 2x^2 + 5x - 6$ ,  $y = 2x^2 - 8x + 15$ ,  $x = 1$  and  $x = 3$ .

(C15.6)

**Resources/Notes**

**Authorized Resource**

*Single Variable Essential Calculus  
Second Edition*

**7.1: Areas Between Curves**

SB: pp. 371 - 375

IG: pp. 375 - 379

PPL: SteEC\_07\_01.ppt

Video Examples: 2

### Calculus Reasoning

#### Outcomes

Students will be expected to

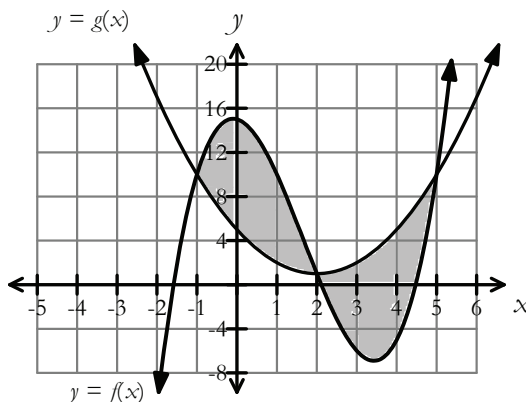
C15 Continued...

#### Achievement Indicator:

C15.6 Continued

#### Elaborations – Strategies for Learning and Teaching

Ask students to determine the total area between two functions such as  $f(x) = x^3 - 5x^2 - x + 15$  and  $g(x) = x^2 - 4x + 5$ .



Discuss with students the following features:

(i) Since  $f(x) \geq g(x)$  on the interval  $[-1, 2]$ , the area is represented by the expression:

$$\int_{-1}^2 [f(x) - g(x)] dx$$

(ii) Since  $g(x) \geq f(x)$  on the interval  $[2, 5]$ , the area is represented by the expression:

$$\int_2^5 [g(x) - f(x)] dx$$

(iii) Since there is a different upper curve in each of the two regions, students must work with the two regions separately. The area they are looking for is the total area bounded by  $f(x)$  and  $g(x)$ , which is given by the area of the region between the curves in the interval  $[-1, 2]$  plus the area of the region between the curves in the interval  $[2, 5]$ .

The following prompts could be used to guide students through the process:

- Find the  $x$ -values of the points of intersection of the two functions by setting  $f(x) = g(x)$ .
- What is the area on the interval  $[-1, 2]$ ?
- What is the area on the interval  $[2, 5]$ ?
- What is the total area?

**General Outcome: Develop introductory calculus reasoning.****Suggested Assessment Strategies***Paper and Pencil*

- Ask students to determine the area of the region between the given curves:

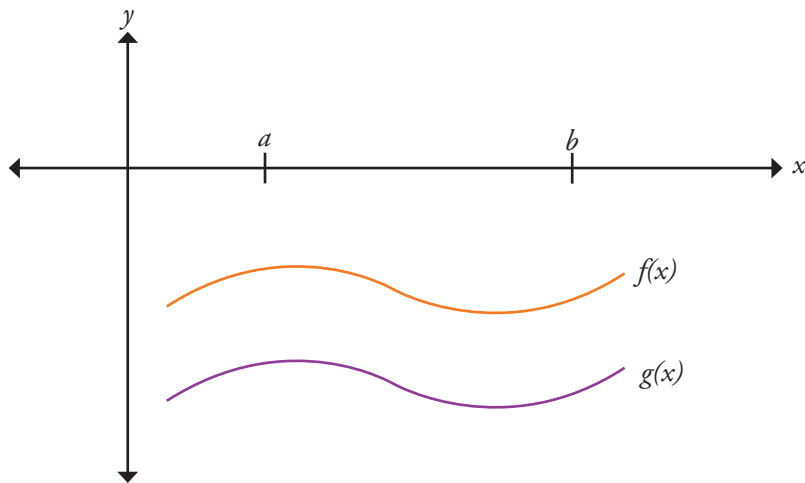
(i)  $y = 2 + x^2$  and  $y = 2x^2 - 7$

(ii)  $y = x^3 + 8$  and  $y = 4x + 8$

(C15.6)

*Interview*

- When determining the area between two negatively valued curves, ask students to explore why  $\int_a^b (f(x) - g(x)) dx$  results in a positive area between the curves.



(C15.6)

**Resources/Notes****Authorized Resource***Single Variable Essential Calculus  
Second Edition***7.1: Areas Between Curves**

SB: pp. 371 - 375

IG: pp. 375 - 379

PPL: SteEC\_07\_01.ppt

Video Examples: 2





## *Appendix A:*

Outcomes with Achievement Indicators  
Organized by Topic  
(With Curriculum Guide References)

<b>Topic:</b> Relations and Functions	<b>General Outcome:</b> Develop algebraic and graphical reasoning through the study of relations.	
<b>Specific Outcomes</b>	<b>Achievement Indicators</b>	<b>Page Reference</b>
It is expected that students will:	<i>The following sets of indicators help determine whether students have met the corresponding specific outcome</i>	
<b>RF1.</b> Demonstrate an understanding of operations on, and compositions of, functions. [CN, R, T, V]	RF1.1 Write a function $h(x)$ as the sum, difference, product or quotient of two or more functions. RF1.2 Sketch the graph of a function that is the sum, difference, product or quotient of two functions, given their graphs. RF1.3 Determine the domain and range of a function that is the sum, difference, product or quotient of two functions. RF1.4 Determine the value of the composition of functions when evaluated at a point, including: <ul style="list-style-type: none"> <li>• <math>f(f(a))</math></li> <li>• <math>f(g(a))</math></li> <li>• <math>g(f(a))</math>.</li> </ul> RF1.5 Determine the equation of the composite function given the equations of two functions $f(x)$ and $g(x)$ : <ul style="list-style-type: none"> <li>• <math>f(f(x))</math></li> <li>• <math>f(g(x))</math></li> <li>• <math>g(f(x))</math>.</li> </ul> RF1.6 Sketch the graph of the composite function given the equations of two functions $f(x)$ and $g(x)$ and determine the domain and range. RF1.7 Determine the original functions from a composition.	pp. 22-24  pp. 22-24  pp. 22-24  p. 26  p. 26  p. 26  p. 26

<b>Topic:</b> Relations and Functions	<b>General Outcome:</b> Develop algebraic and graphical reasoning through the study of relations.	
<b>Specific Outcomes</b> It is expected that students will:	<b>Achievement Indicators</b> <i>The following sets of indicators help determine whether students have met the corresponding specific outcome</i>	<b>Page Reference</b>
<b>RF2.</b> Graph and analyze rational functions (limited to numerators and denominators that are that are polynomials of degree $\leq 3$ ). [CN, R, T, V]	RF2.1 Explain the behaviour of the graph of a rational function for values of the variable near a non-permissible value.	pp. 66-72
	RF2.2 Determine if the graph of a rational function will have an asymptote or a hole for a non-permissible value.	pp. 66-72
	RF2.3 Sketch the graph of a rational function.	pp. 66-72

Topic: Calculus Reasoning	General Outcome: Develop introductory calculus reasoning.	
Specific Outcomes	Achievement Indicators	Page Reference
It is expected that students will:	<i>The following sets of indicators help determine whether students have met the corresponding specific outcome</i>	
<p>C1. Demonstrate an understanding of the concept of limit and evaluate the limit of a function.</p> <p>[C, CN, R, T, V]</p>	<p>C1.1 Using informal methods, explore the concept of a limit including one sided limits.</p> <p>C1.2 Using informal methods, establish that the limit of <math>\frac{1}{x}</math> as <math>x</math> approaches infinity is zero.</p> <p>C1.3 Explore the concept of limit and the notation used in expressing the limit of a function:</p> <ul style="list-style-type: none"> <li>• <math>\lim_{x \rightarrow a^+} f(x)</math></li> <li>• <math>\lim_{x \rightarrow a^-} f(x)</math></li> <li>• <math>\lim_{x \rightarrow a} f(x)</math>.</li> </ul> <p>C1.4 Determine the value of the limit of a function as the variable approaches a real number</p> <ul style="list-style-type: none"> <li>• by using a provided graph, including piecewise functions</li> <li>• by using a table of values.</li> </ul> <p>C1.5 Evaluate one-sided limits using a graph.</p> <p>C1.6 Apply the properties of limits including:</p> <ul style="list-style-type: none"> <li>• Sum Rule</li> <li>• Difference Rule</li> <li>• Product Rule</li> <li>• Constant Multiple Rule</li> <li>• Quotient Rule</li> <li>• Power Rule</li> </ul> <p>to solve problems.</p> <p>C1.7 Determine the value of the limit of a function as the variable approaches a real number</p> <ul style="list-style-type: none"> <li>• by substitution</li> <li>• by algebraic manipulation.</li> </ul> <p>C1.8 Determine limits that result in infinity (infinite limits).</p> <p>C1.9 Investigate the behaviour of the function at a vertical asymptote using limits.</p> <p>C1.10 Evaluate limits of functions as <math>x</math> approaches infinity (limits at infinity).</p> <p>C1.11 Investigate the end behaviour of the function using limits to identify possible horizontal asymptotes.</p> <p>C1.12 Investigate the end behaviour of the function using limits to identify possible oblique asymptotes.</p>	<p>pp. 32-34</p> <p>p. 34</p> <p>p. 36</p> <p>pp. 36-40</p> <p>pp. 36-40</p> <p>pp. 40-44</p> <p>pp. 40-44</p> <p>pp. 44-48</p> <p>pp. 44-48</p> <p>pp. 48-50</p> <p>pp. 48-50</p> <p>p. 52</p>

Topic: Calculus Reasoning	General Outcome: Develop introductory calculus reasoning.	
Specific Outcomes	Achievement Indicators	Page Reference
It is expected that students will:	<i>The following sets of indicators help determine whether students have met the corresponding specific outcome</i>	
C2. Solve problems involving continuity. [C, CN, R, T, V]	<p>C2.1 Distinguish between the concepts of continuity and discontinuity of a function informally.</p> <p>C2.2 Identify examples of discontinuous functions and the types of discontinuities they illustrate, such as removable, infinite, jump, and oscillating discontinuities.</p> <p>C2.3 Determine whether a function is continuous at a point from its graph.</p> <p>C2.4 Determine whether a function is continuous at a point using the definition of continuity.</p> <p>C2.5 Determine whether a function is continuous on a closed interval.</p> <p>C2.6 Rewrite removable discontinuities by extending or modifying a function.</p>	<p>p. 54</p> <p>p. 54</p> <p>pp. 56-58</p> <p>pp. 56-58</p> <p>p. 58</p> <p>p. 60</p>

Topic: Calculus Reasoning	General Outcome: Develop introductory calculus reasoning.	
Specific Outcomes	Achievement Indicators	Page Reference
It is expected that students will:	<i>The following sets of indicators help determine whether students have met the corresponding specific outcome</i>	
<p>C3. Demonstrate an understanding of the concept of a derivative and evaluate derivatives of functions using the definition of derivative.</p> <p>[CN, ME, R, V]</p>	<p>C3.1 Describe geometrically a secant line and a tangent line for the graph of a function at <math>x = a</math>.</p> <p>C3.2 Determine the average rate of change of a function over an interval.</p> <p>C3.3 Identify the instantaneous rate of change of a function at a point as the limiting value of a sequence of average rates of change.</p> <p>C3.4 Define and evaluate the derivative at <math>x = a</math> as:  <math display="block">\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \text{ and } \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} .</math></p> <p>C3.5 Determine the equation of the tangent line and normal line to a graph of a relation at a given point.</p> <p>C3.6 Define and determine the derivative of a function using  <math display="block">f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}</math>           (limited to polynomials of degree 3, square root and rational functions with linear terms).</p> <p>C3.7 Use alternate notation interchangeably to express derivatives (<i>i.e.</i>, <math>f'(x)</math>, <math>\frac{dy}{dx}</math>, <math>y'</math> etc.).</p> <p>C3.8 Determine whether a function is differentiable at a given point.</p> <p>C3.9 Explain why a function is not differentiable at a given point, and distinguish between corners, cusps, discontinuities, and vertical tangents.</p> <p>C3.10 Determine all values for which a function is differentiable, given the graph.</p> <p>C3.11 Sketch a graph of the derivative of a function, given the graph of a function.</p> <p>C3.12 Sketch a graph of the function, given the graph of the derivative of a function.</p>	<p>pp. 78-80</p> <p>pp. 78-80</p> <p>pp. 78-80</p> <p>pp. 82-86</p> <p>p. 86</p> <p>pp. 86-88</p> <p>p. 90</p> <p>pp. 90-94</p> <p>pp. 90-94</p> <p>pp. 90-94</p> <p>p. 96</p> <p>p. 96</p>

Topic: Calculus Reasoning	General Outcome: Develop introductory calculus reasoning.	
Specific Outcomes	Achievement Indicators	Page Reference
It is expected that students will:	<i>The following sets of indicators help determine whether students have met the corresponding specific outcome</i>	
<p>C4. Apply derivative rules including:</p> <ul style="list-style-type: none"> <li>• Constant Rule</li> <li>• Constant Multiple Rule</li> <li>• Sum Rule</li> <li>• Difference Rule</li> <li>• Product Rule</li> <li>• Quotient Rule</li> <li>• Power Rule</li> <li>• Chain Rule</li> </ul> <p>to determine the derivatives of functions. [C, CN, PS, R]</p>	<p>C4.1 Derive the Constant, Constant Multiple, Sum, Difference, Product, and Quotient Rules for determining derivatives.</p> <p>C4.2 Determine derivatives of functions, using the Constant, Constant Multiple, Power, Sum, Difference, Product and Quotient Rules.</p> <p>C4.3 Determine second and higher-order derivatives of functions.</p> <p>C4.4 Determine derivatives of functions using the Chain Rule.</p> <p>C4.5 Solve problems involving derivatives drawn from a variety of applications, limited to tangent and normal lines, straight line motion and rates of change.</p>	<p>pp. 98-104</p> <p>p. 104</p> <p>p. 106</p> <p>pp. 106-110</p> <p>pp. 110-114</p>
<p>C5. Determine the derivative of a relation, using implicit differentiation. [C, CN, PS, R, V]</p>	<p>C5.1 Determine the derivative of an implicit relation.</p> <p>C5.2 Determine the equation of the tangent and normal line to the graph of a relation at a given point.</p> <p>C5.3 Determine the second derivative of a relation, using implicit differentiation.</p>	<p>p. 116</p> <p>p. 118</p> <p>p. 118</p>

Topic: Calculus Reasoning	General Outcome: Develop introductory calculus reasoning.	
Specific Outcomes	Achievement Indicators	Page Reference
It is expected that students will:	<i>The following sets of indicators help determine whether students have met the corresponding specific outcome</i>	
<p>C6. Use derivatives to sketch the graph of a polynomial function. [C, CN, PS, R, T, V]</p>	<p>C6.1 Use <math>f'(x)</math> to identify the critical numbers, relative and absolute extrema, and intervals of increase and decrease.</p> <p>C6.2 Use <math>f''(x)</math> to identify the hypercritical numbers, points of inflection and intervals of concavity.</p> <p>C6.3 Sketch the graph of <math>f(x)</math> using information obtained from the function and its derivatives.</p> <p>C6.4 Use the given function <math>f(x)</math> to determine its features such as intercepts and the domain.</p>	<p>pp. 124-132</p> <p>pp. 134-138</p> <p>p. 140</p> <p>p. 140</p>
<p>C7. Use derivatives to sketch the graph of a rational function. [C, CN, PS, R, T, V]</p>	<p>C7.1 Use <math>f'(x)</math> to identify the critical numbers, relative and absolute extrema and intervals of increase and decrease.</p> <p>C7.2 Use <math>f''(x)</math> to identify the hypercritical numbers, points of inflection and intervals of concavity.</p> <p>C7.3 Sketch the graph of <math>f(x)</math> using information obtained from the function and its derivatives.</p> <p>C7.4 Use the given function <math>f(x)</math> to determine its features such as intercepts, asymptotes, points of discontinuity and the domain.</p>	<p>pp. 142-144</p> <p>pp. 142-144</p> <p>pp. 142-144</p> <p>pp. 142-144</p>
<p>C8. Use calculus techniques to solve and interpret related rates problems. [C, CN, ME, PS, R]</p>	<p>C8.1 Solve a problem involving related rates drawn from a variety of applications.</p> <p>C8.2 Interpret the solution to a related rates problem.</p>	<p>pp. 146-148</p> <p>pp. 146-148</p>



Topic: Calculus Reasoning	General Outcome: Develop introductory calculus reasoning.	
Specific Outcomes	Achievement Indicators	Page Reference
It is expected that students will:	<i>The following sets of indicators help determine whether students have met the corresponding specific outcome</i>	
<b>C9.</b> Use calculus techniques to solve optimization problems. [C, CN, ME, PS, R]	C9.1 Determine the equation of the function to be optimized in an optimization problem.  C9.2 Determine the equations of any parameters necessary in an optimization problem.  C9.3 Solve an optimization problem drawn from a variety of applications, using calculus techniques.  C9.4 Interpret the solution(s) to an optimization problem.	p. 150  p. 150  p. 150  p. 150
<b>C10.</b> Solve problems involving limits of trigonometric functions. [C, CN, ME, PS R, T, V]	C10.1 Establish each of the following trigonometric limits, using informal methods: <ul style="list-style-type: none"> <li>• <math>\lim_{x \rightarrow 0} \sin x = 0</math></li> <li>• <math>\lim_{x \rightarrow 0} \cos x = 1</math></li> <li>• <math>\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1</math></li> <li>• <math>\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = 0</math> .</li> </ul> C10.2 Evaluate limits involving trigonometric expressions.	p. 156          pp. 158-160
<b>C11.</b> Apply derivatives of trigonometric functions. [CN, ME, R, V]	C11.1 Derive the derivatives of sine, cosine, tangent, cotangent, secant and cosecant.  C11.2 Determine the derivative of expressions involving trigonometric functions.  C11.3 Solve problems involving the derivative of a trigonometric function.	pp. 162-166  p. 166  pp. 168-170

Topic: Calculus Reasoning	General Outcome: Develop introductory calculus reasoning.	
Specific Outcomes	Achievement Indicators	Page Reference
It is expected that students will:	<i>The following sets of indicators help determine whether students have met the corresponding specific outcome</i>	
<b>C12.</b> Solve problems involving inverse trigonometric functions. [CN, ME, R, V]	C12.1 Explain the relationship between the trigonometric functions and their inverses. C12.2 Explain why trigonometric functions must have their domains restricted to create inverse trigonometric functions. C12.3 Sketch the graph of an inverse trigonometric function. C12.4 Determine the exact value of an expression involving an inverse trigonometric function. C12.5 Derive the inverse trigonometric derivatives. C12.6 Determine the derivative of an inverse trigonometric function. C12.7 Solve problems involving the derivative of an inverse trigonometric function.	p. 172  p. 172  p. 174 p. 176  pp. 176-182  p. 182  p. 182
<b>C13.</b> Determine the limit and derivative of exponential, logarithmic, and hyperbolic trigonometric functions. [CN, ME, R, V]	C13.1 Graph and analyze exponential and logarithmic functions $y = e^x$ and $y = \ln x$ . C13.2 Establish the exponential limit $\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$ using informal methods. C13.3 Derive the derivatives of the exponential functions $e^x$ and $a^x$ , and the logarithmic functions $\ln x$ and $\log_a x$ . C13.4 Determine the derivative of an exponential function.  C13.5 Solve problems involving the derivative of an exponential or a logarithmic function. C13.6 Determine the derivative of a logarithmic function. C13.7 Determine the derivative of a function using logarithmic differentiation. C13.8 Define and characterize the hyperbolic trigonometric functions $\sinh x$ and $\cosh x$ in terms of the exponential function $e$ . C13.9 Define $\tanh x$ , $\operatorname{csch} x$ , $\operatorname{coth} x$ , and $\operatorname{sech} x$ in terms of $\sinh x$ and $\cosh x$ . C13.10 Determine the derivative of a hyperbolic trigonometric function.	p. 188  p. 190  pp. 190-196  p. 192  pp. 192-194 pp. 194-196  p. 198  p. 200  p. 200  pp. 200-202

Topic: Calculus Reasoning	General Outcome: Develop introductory calculus reasoning.	
Specific Outcomes	Achievement Indicators	Page Reference
It is expected that students will:	<i>The following sets of indicators help determine whether students have met the corresponding specific outcome</i>	
<p><b>C14.</b> Determine the indefinite integral of polynomial and radical functions.</p> <p>[C, CN, PS, R]</p>	<p>C14.1 Explain the meaning of the phrase “<math>F(x)</math> is an antiderivative of <math>f(x)</math>”.</p> <p>C14.2 Determine the general antiderivative of functions.</p> <p>C14.3 Use antiderivative notation appropriately (<i>i.e.</i>, <math>\int f(x)dx</math> for the antiderivative of <math>f(x)</math>).</p> <p>C14.4 Identify the properties of antidifferentiation.</p> <p>C14.5 Determine the indefinite integral of a function given extra conditions.</p> <p>C14.6 Use antidifferentiation to solve problems about motion of a particle along a line that involves:</p> <ul style="list-style-type: none"> <li>• computing the displacement given the initial position and velocity as a function of time</li> <li>• computing velocity and/or displacement given the suitable initial conditions and acceleration as a function of time.</li> </ul>	<p>p. 208</p> <p>pp. 208-212</p> <p>p. 210</p> <p>pp. 210-212</p> <p>p. 214</p> <p>pp. 214-216</p>
<p><b>C15.</b> Determine the definite integral of a polynomial function.</p> <p>[C, CN, PS, R]</p>	<p>C15.1 Estimate an area using a finite sum.</p> <p>C15.2 Determine the area using the infinite Riemann sum.</p> <p>C15.3 Convert a Riemann sum to a definite integral.</p> <p>C15.4 Using definite integrals, determine the area under a polynomial function from <math>x = a</math> to <math>x = b</math>.</p> <p>C15.5 Calculate the definite integral of a function over an interval <math>[a, b]</math>.</p> <p>C15.6 Determine the area between two polynomial functions.</p>	<p>pp. 218-224</p> <p>pp. 218-224</p> <p>p. 224</p> <p>pp. 226-228</p> <p>pp. 226-228</p> <p>pp. 230-232</p>



## *Appendix B:*

### Further Elaboration on Riemann Sums

In this appendix, Riemann sums with an infinite number of rectangles will be examined by evaluating

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x.$$

Students were exposed to  $S_n = \sum_{i=1}^n f(x_i) \Delta x$  as the sum of  $n$  rectangles with dimensions  $f(x_i)$  by  $\Delta x$  on an

interval. It was also stated that  $\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$ .

Sigma (or summation) notation is used to express sums in a compact way. The sum of the first twenty positive integers  $1 + 2 + 3 + \dots + 20$  can be represented using sigma notation as  $\sum_{i=1}^{20} i$ , where the variable  $i$  is the index of summation.

Students should use the following properties of summation to evaluate the sum.

$$\sum_{i=1}^n c = cn \qquad \sum_{i=1}^n i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^n i^3 = \left[ \frac{n(n+1)}{2} \right]^2$$

It is not the intent to derive the summation properties. It is, however, beneficial for students to use a numerical example to help them visualize the workings for the sum of a constant series and the sum of a linear series.

Ask students to first consider the sum of a constant series. In a constant series, such as  $3 + 3 + 3 + 3 + 3$ , each term has the same value.

$$\sum_{i=1}^5 3 = \underbrace{3 + 3 + 3 + 3 + 3}_{5 \text{ times}} = 3 \cdot 5 = 15$$

The formula for the sum of a constant series is  $\sum_{i=1}^n c = cn$  as shown.

$$\sum_{i=1}^n c = \underbrace{c + c + c + \dots + c}_{n \text{ times}} = n \cdot c = cn$$

A linear series is a counting series, such as the sum of the first 10 natural numbers. Ask students to examine the pattern when the terms are rearranged.

$$\begin{aligned} \sum_{i=1}^{10} i &= 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 \\ &= (1+10) + (2+9) + (3+8) + (4+7) + (5+6) \\ &= 11 + 11 + 11 + 11 + 11 \\ &= 5(11) \end{aligned}$$

Students should notice there are 5 pairs, each with a sum of 11, which leads to  $\sum_{i=1}^{10} i = \frac{10}{2}(1+10)$ .

Working with Riemann sums can be time consuming when there is a large number of subintervals. Discuss with students that, with sigma notation, a Riemann sum provides a convenient compact form.

$$f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x = \sum_{i=1}^n f(x_i)\Delta x$$

Using the right Riemann sum in sigma notation, students must identify the point  $x_i$ . The notation is written as  $x_i = a + i\Delta x$ , where  $i = 1, 2, 3, \dots, n$ .

In the Antidifferentiation and Integration unit, students used graphing technology to approximate the area under the curve  $y = x^2$  over the interval  $[0, 3]$  for 100 subintervals to be 9.13545 (C14.1, 14.2). Ask students to verify this result using sigma notation and the summation properties:

$$\Delta x = 0.03 \text{ and } x_i = 0 + i(0.03) = 0.03i$$

$$\begin{aligned} \sum_{i=1}^{100} f(x_i)\Delta x &= \sum_{i=1}^{100} (0.03i)^2 0.03 \\ &= \sum_{i=1}^{100} (0.0009i^2)0.03 \\ &= 0.000027 \sum_{i=1}^{100} i^2 \\ &= (0.000027) \sum_{i=1}^{100} \frac{n(n+1)(2n+1)}{6} \\ &= (0.000027) \frac{(100)(101)(201)}{6} \\ &= 9.13545 \end{aligned}$$

Students should then progress to evaluating the Riemann sum for arbitrary values of  $n$  where they evaluate the limit of the Riemann sum. In general, if  $f(x)$  is integrable on  $[a, b]$ , then the area under the curve can be summarized as:

$$\text{Area} = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i)\Delta x \text{ where } \Delta x = \frac{b-a}{n} \text{ and } x_i = a + i\Delta x$$

Students should continue to work with the previous example, where they approximated the area under the curve  $y = x^2$  over the interval  $[0, 3]$ . They will now use limits and the summation properties to verify that the approximation approaches 9. Guide students through the problem:

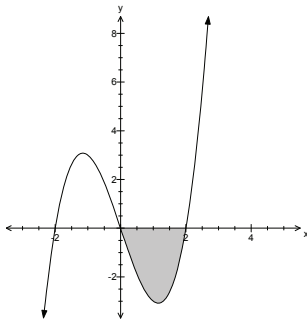
- What is the width of each rectangle if the interval  $[0, 3]$  is divided into  $n$  rectangles? ( $\Delta x = \frac{b-a}{n} = \frac{3}{n}$ )
- What are the right endpoints? ( $x_i = a + i\Delta x = 0 + i(\frac{3}{n}) = \frac{3i}{n}$ )
- Write an expression for the area of the  $i^{\text{th}}$  rectangle. ( $f(x_i)\Delta x = f(\frac{3i}{n})(\frac{3}{n})$ )
- Determine the Riemann sum using  $n$  subintervals.

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{9i^2}{n^2} \left(\frac{3}{n}\right) \\ &= \lim_{n \rightarrow \infty} \frac{27}{n^3} \sum_{i=1}^n i^2 \\ &= \lim_{n \rightarrow \infty} \frac{27}{n^3} \left(\frac{n(n+1)(2n+1)}{6}\right) \\ &= 9 \end{aligned}$$

When working with the Riemann sum, include examples where the interval does not start at  $x = 0$ . For the previous example, ask students how the right endpoint and the width of the rectangle would change if the interval was from  $[1, 3]$ .

So far, students have worked through examples where the function  $f(x)$  was positive on the interval. They should now investigate the geometric meaning of Riemann sums when  $f(x)$  is negative on some or all of the interval  $[a, b]$ .

- Ask students to evaluate the Riemann sum for a function, such as  $f(x) = x^3 - 4x$ , on the interval  $[0, 2]$  with  $n$  equally spaced subintervals.



$$\begin{aligned} & \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f\left(\frac{2i}{n}\right) \frac{2}{n} \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[ \left(\frac{2i}{n}\right)^3 - 4\left(\frac{2i}{n}\right) \right] \frac{2}{n} \\ &= \lim_{n \rightarrow \infty} \left[ \frac{16}{n^4} \sum_{i=1}^n i^3 - \frac{16}{n^2} \sum_{i=1}^n i \right] \\ &= \lim_{n \rightarrow \infty} \left\{ \frac{16}{n^4} \left[ \frac{n(n+1)}{2} \right]^2 - \frac{16}{n^2} \left[ \frac{n(n+1)}{2} \right] \right\} \\ &= 4 - 8 \end{aligned}$$

Students should recognize that they have to use the formula for the sum of cubes and the properties of sums to result in a Riemann sum of  $-4$ . The area is actually  $4$ , and the negative sign is an indication that the area lies below the  $x$ -axis. Discuss with students that this value represents the signed area, where the region located below the  $x$ -axis will be counted as negative, and the region above will be counted as positive.

Questions:

1. Ask students to evaluate the following:

(i)  $\sum_{i=1}^{20} (2i^2 - 3i)$

(ii)  $\sum_{i=1}^n (3i^3 + 2i + 10)$

2. Ask students to use the Riemann sum to answer the following:

(i) Estimate the area under the curve  $y = x^2 + 3$  from  $0 \leq x \leq 1$  using 5 rectangles.

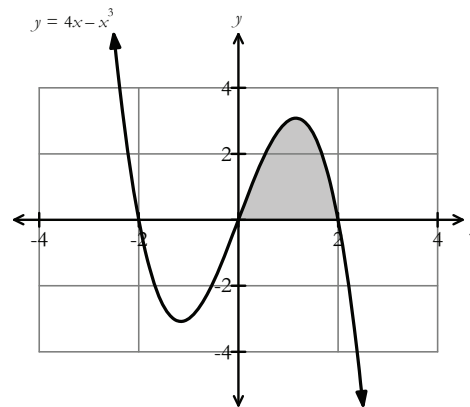
(ii) Estimate the area of the region  $y = (x + 1)^2$  from  $x = 0$  and  $x = 2$  using 6 subintervals.

(iii) Determine the area under  $y = x^3$  from  $x = 1$  and  $x = 4$  using:

- 9 subintervals
- an infinite number of subintervals.

3. Ask students to answer the following:

(i) Use the Riemann sum to estimate the area under the curve  $y = 4x - x^3$  from  $0 \leq x \leq 2$ .



(ii) Why is the value of the Riemann sum positive?



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